EE 505

Lecture 8

Clock Jitter

Statistical Circuit Modeling

Review from last lecture

Windowing - a strategy to address the problem of requiring precisely an integral number of periods to use the DFT for Spectral analysis?

- Windowing is sometimes used
- Windowing is sometimes misused

Review from last lecture

Windowing

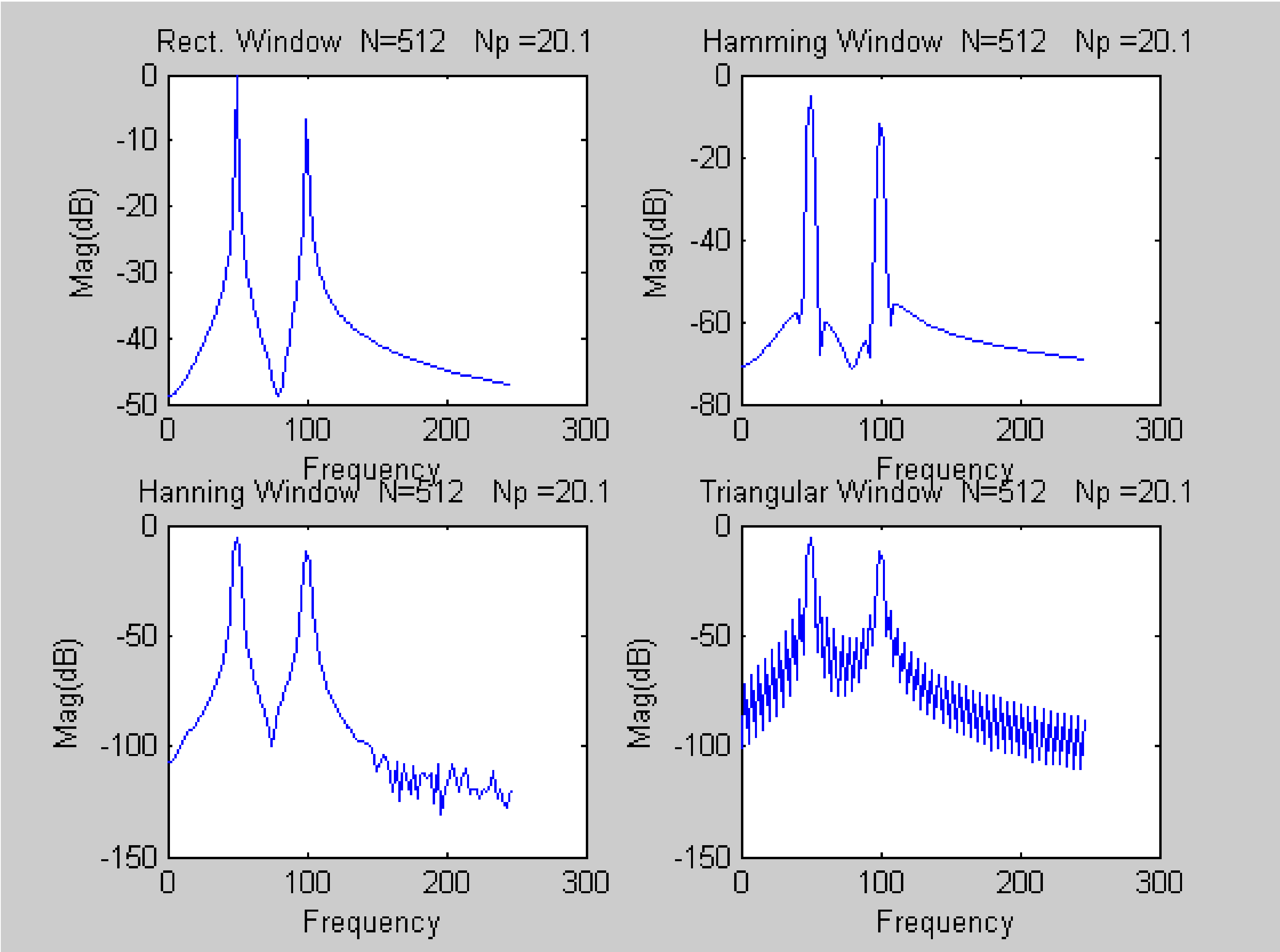
Windowing is the weighting of the time domain function to maintain continuity at the end points of the sample window

Well-studied window functions:

- Rectangular (also with appended zeros)
- Triangular
- Hamming
- Hanning
- Blackman

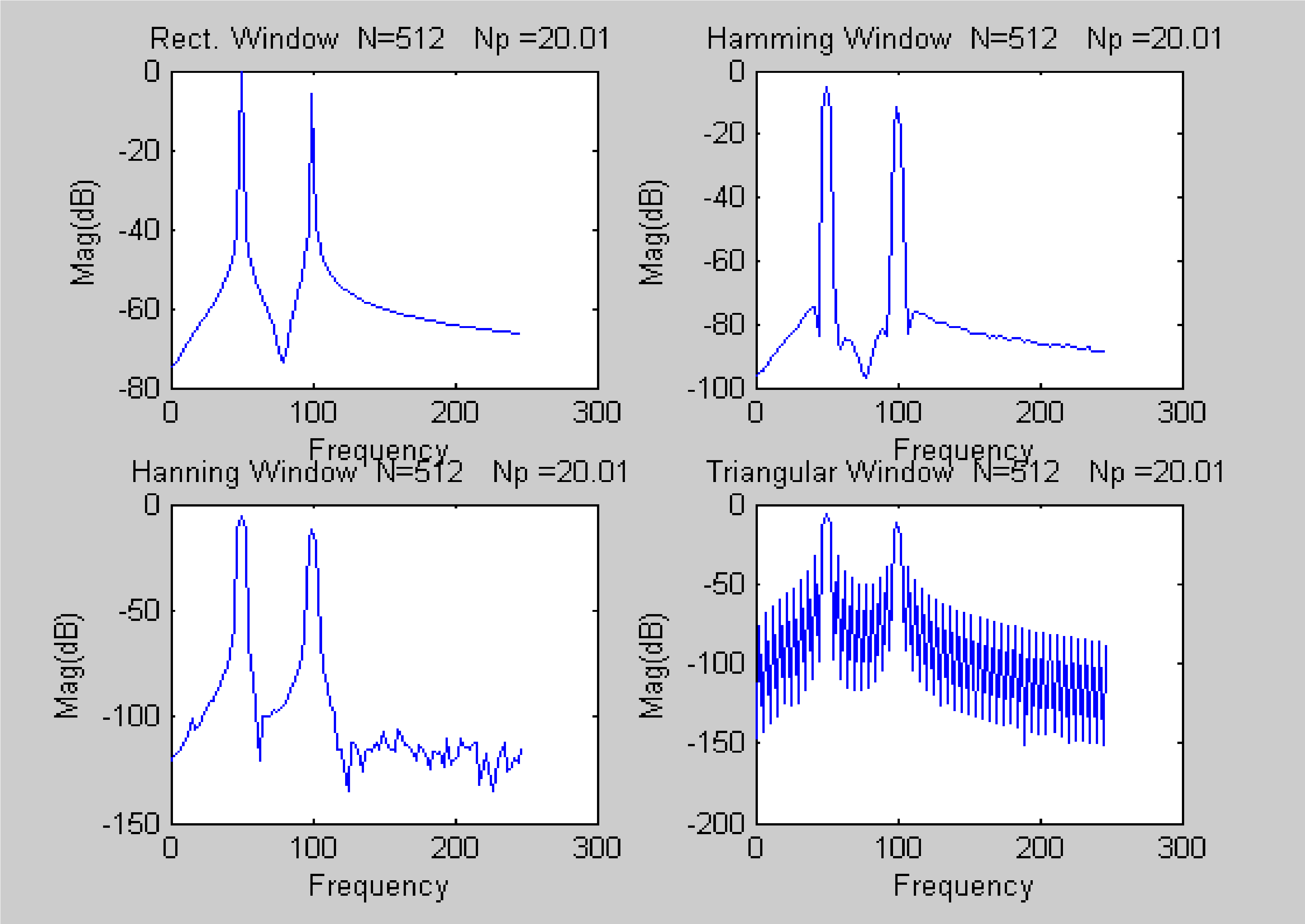
Review from last lecture

Comparison of 4 windows



Review from last lecture

Comparison of 4 windows



Review from last lecture

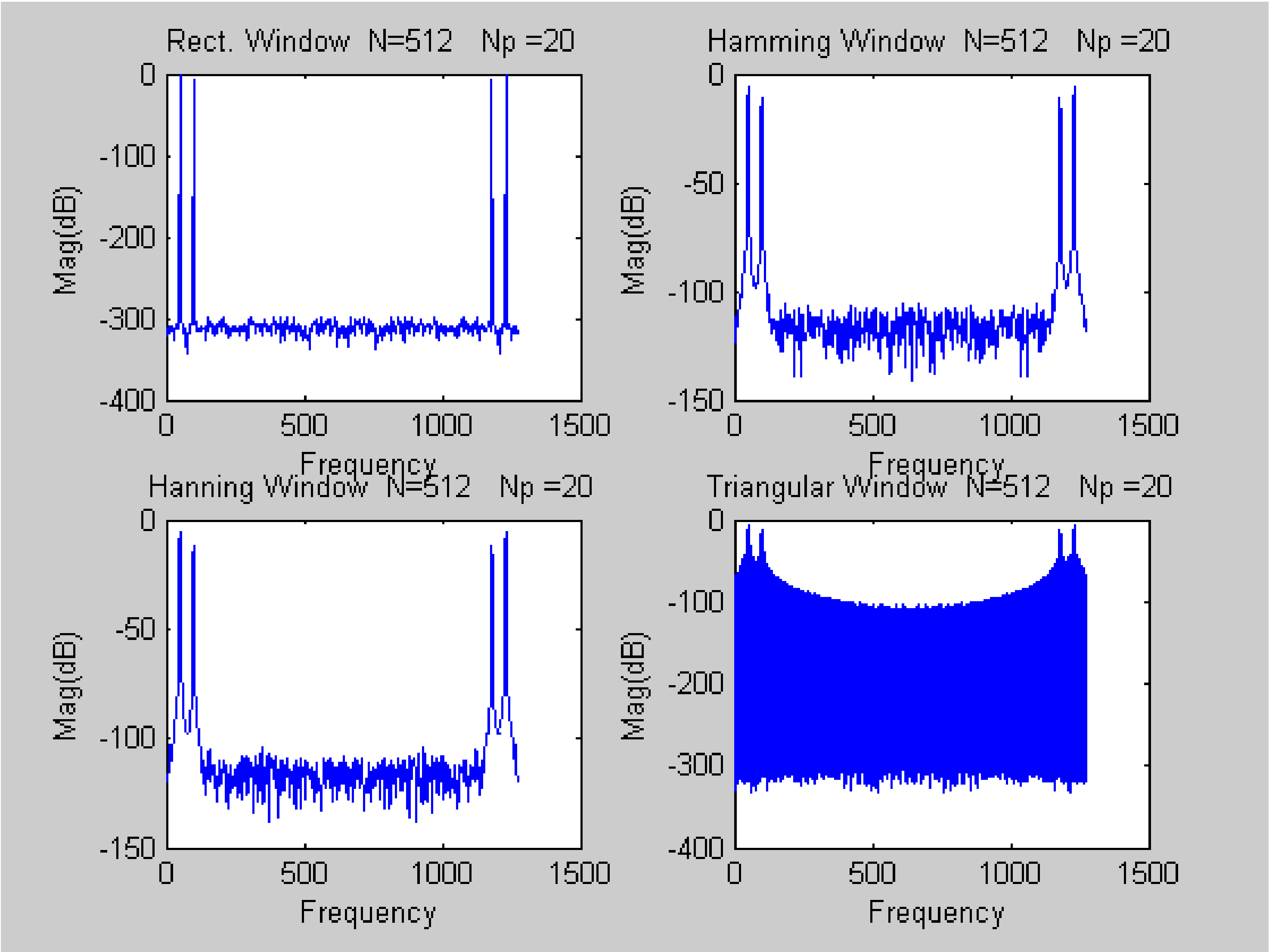
Preliminary Observations about Windows

- Provide separation of spectral components
- Energy can be accumulated around spectral components
- Simple to apply
- Some windows work much better than others

But – windows do not provide dramatic improvement and …

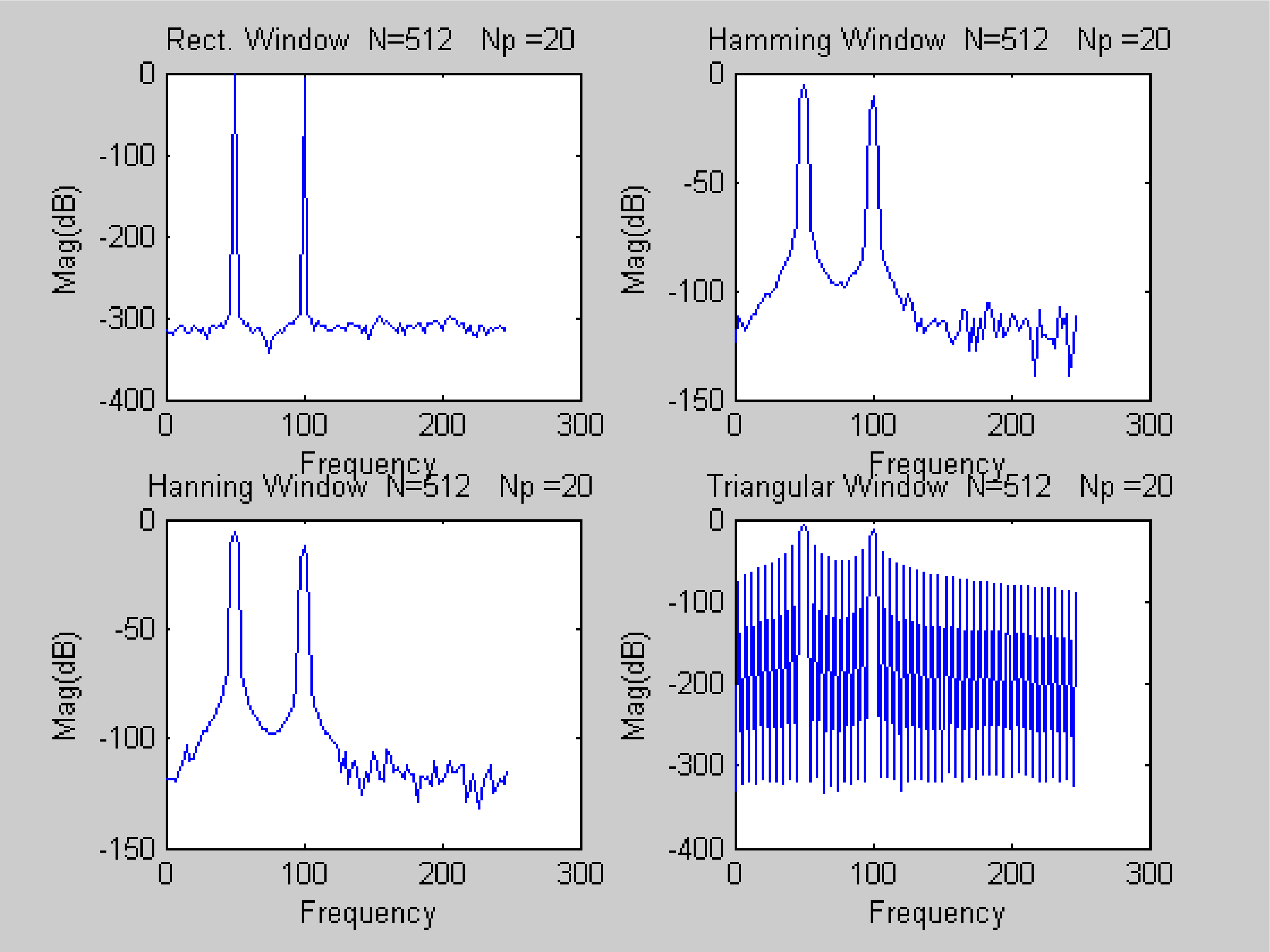
Review from last lecture

Comparison of 4 windows when sampling hypothesis are satisfied



Review from last lecture

Comparison of 4 windows



Review from last lecture

Preliminary Observations about Windows

- Provide separation of spectral components
- Energy can be accumulated around spectral components
- Simple to apply
- Some windows work much better than others

But – windows do not provide dramatic improvement and can significantly degrade performance if sampling hypothesis are met

Review from last lecture

Quantization Effects

time and amplitude depicted

Zero-order sample/hold on DAC or zero-order hold on ADC interpreted output

DAC

Assume DAC will be used to generate a continuous time signal

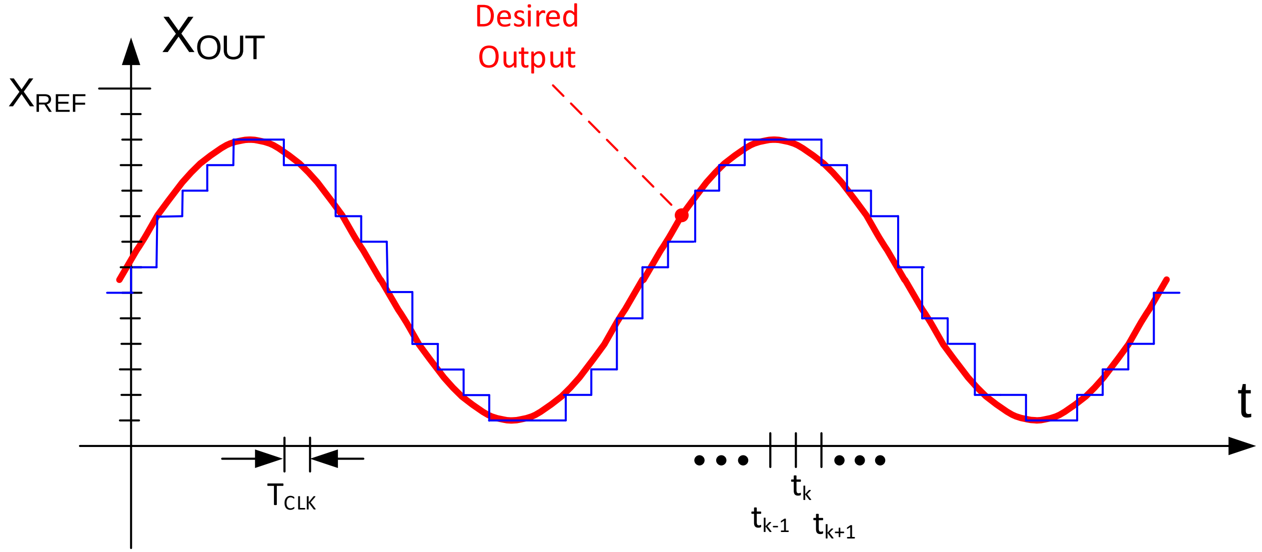
Assume DAC is driven by a clock of period T_{CLK}

DAC inputs will be a discrete sequence $\vec{X}(t_k) = \langle x_{quant}(t_k) \rangle$

DAC inputs can change only at times t_k

The duration of each DAC input depends upon system

With zero-order S/H, it is assumed that the DAC output remains constant between transaction times $x_{OUT}(t) = x_{quant}(t_k) \qquad t_k \le t < t_{k+1}$

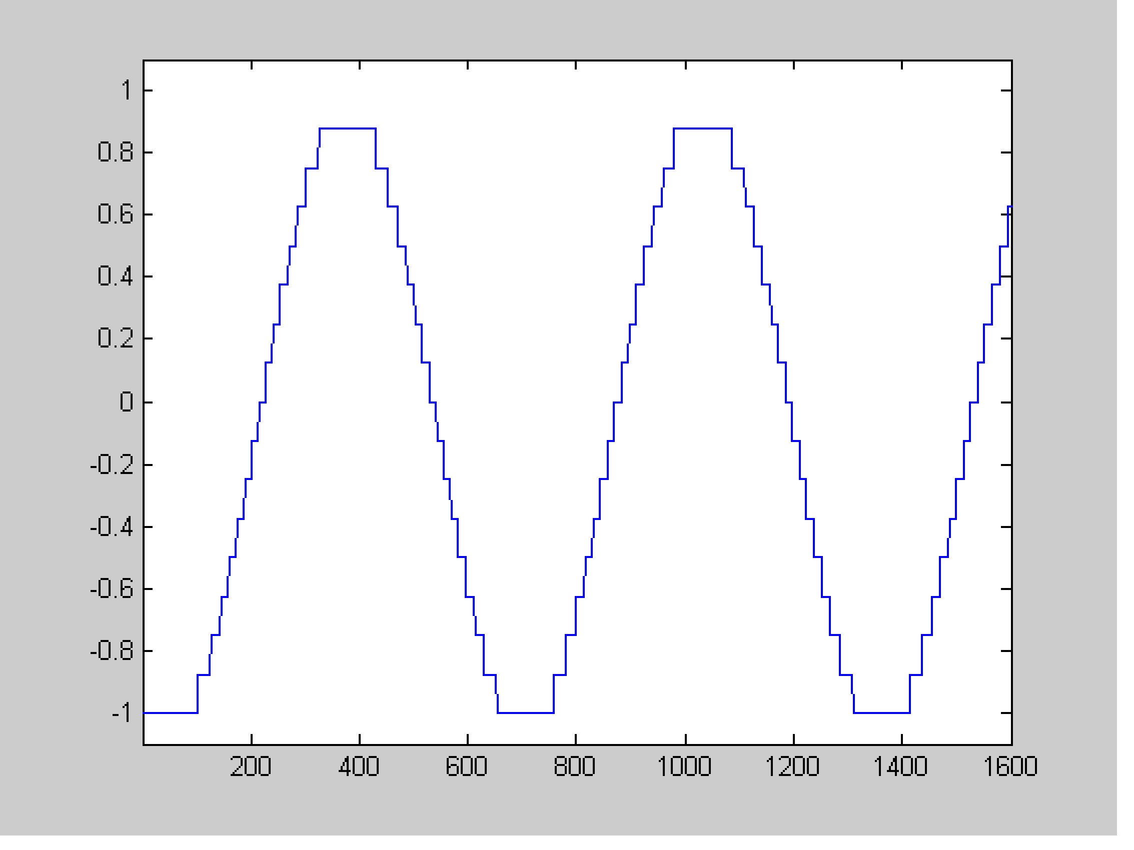


Review from last lecture

Quantization Effects

(time and amplitude depicted)

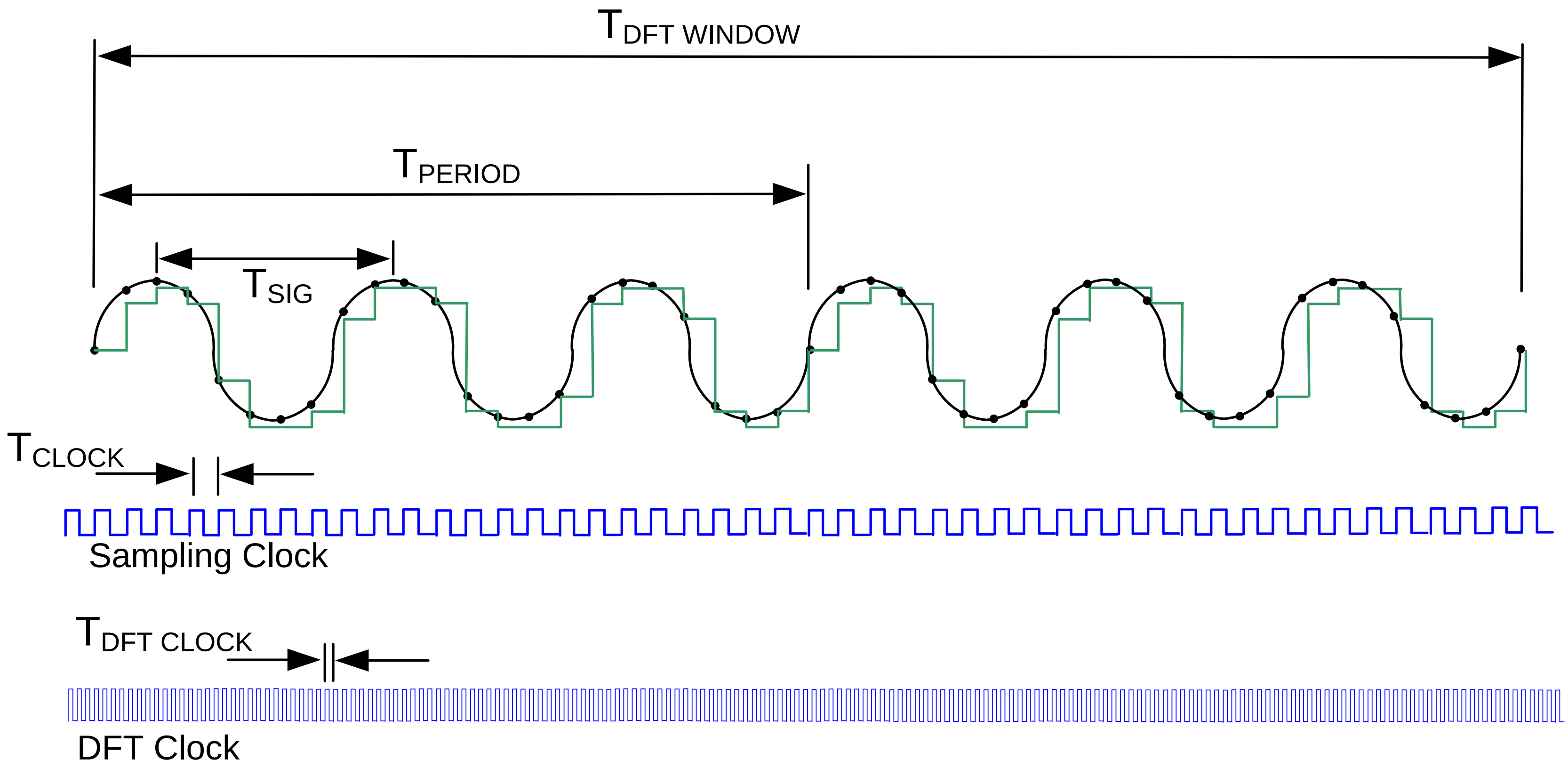
16,384 pts   res = 4bits



Is this signal band limited?

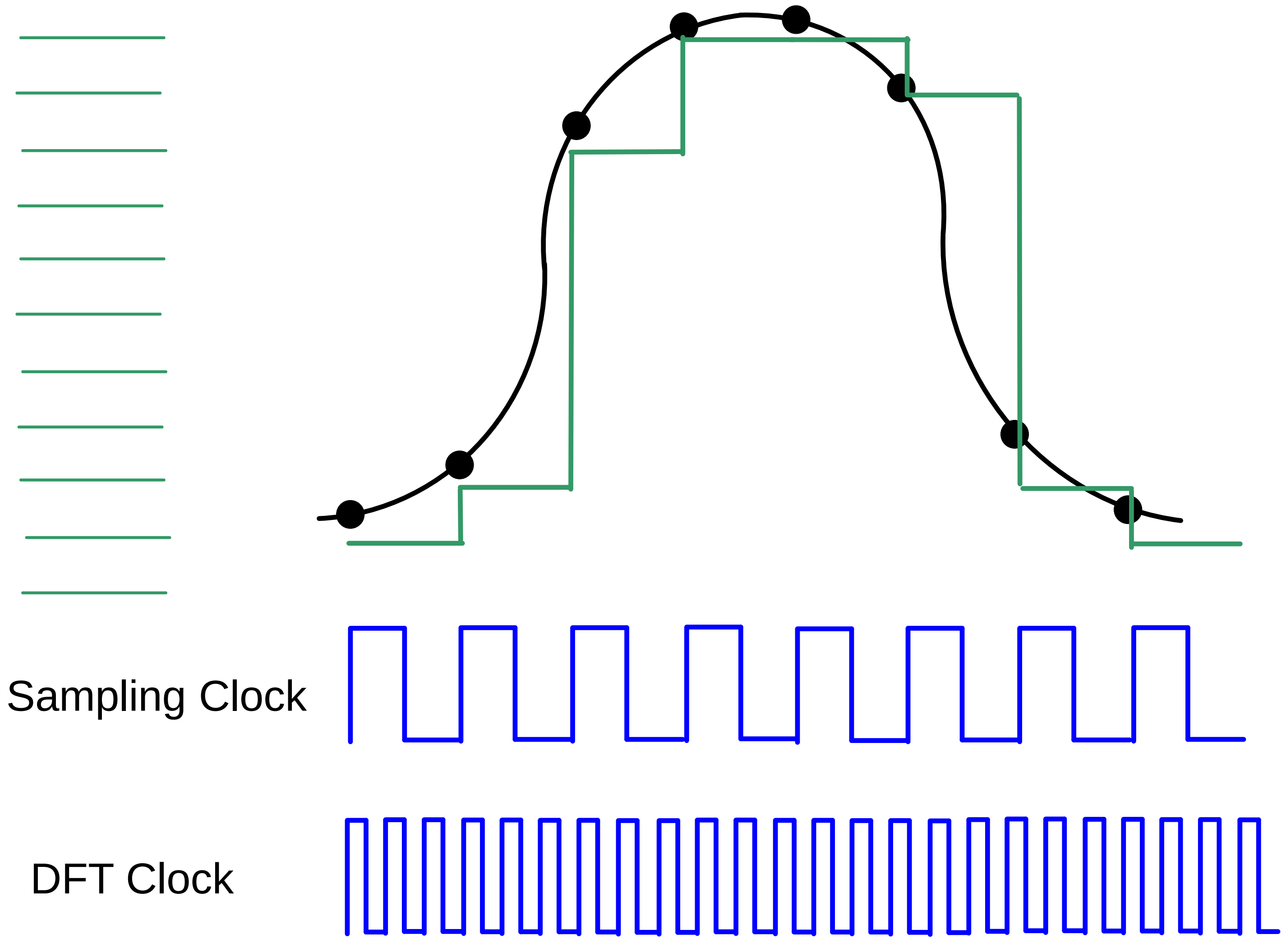
Review from last lecture

Spectral Characteristics of DAC



Review from last lecture

Spectral Characteristics of DAC

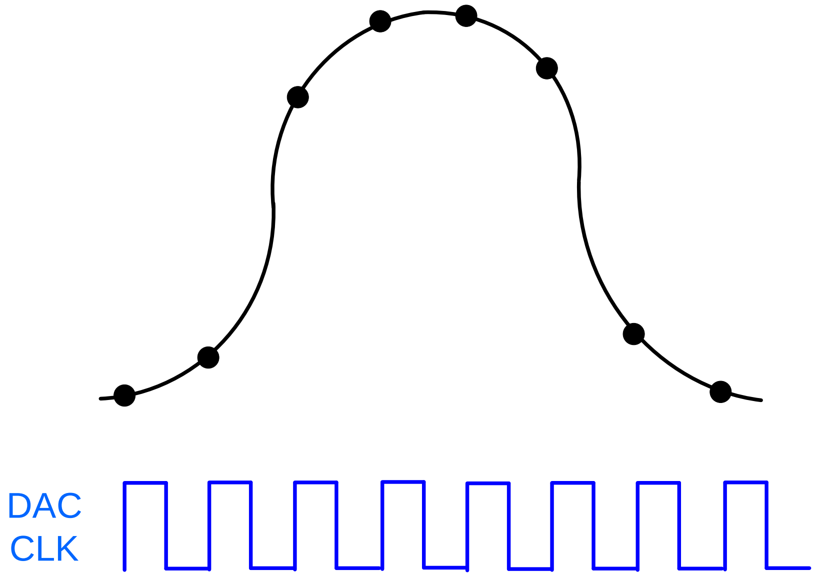


Summary of time and amplitude quantization assessment

Time and amplitude quantization do not introduce harmonic distortion

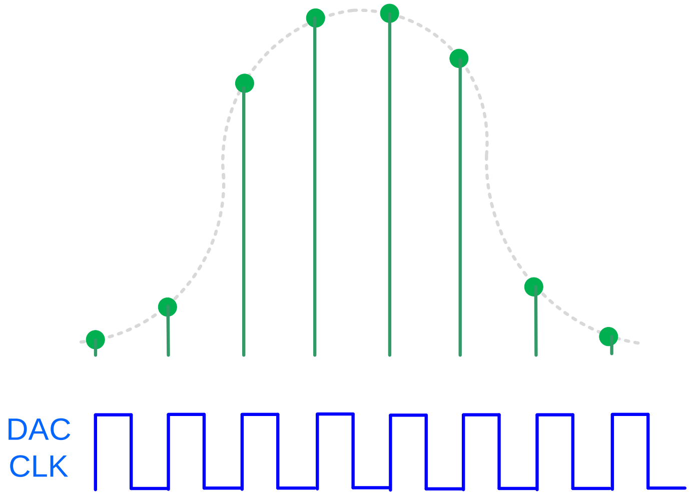
Time and amplitude quantization do increase the noise floor

Duty Cycle Effects on Spectral Performance of DACS (File: DAC Quantization with RTZ.m)

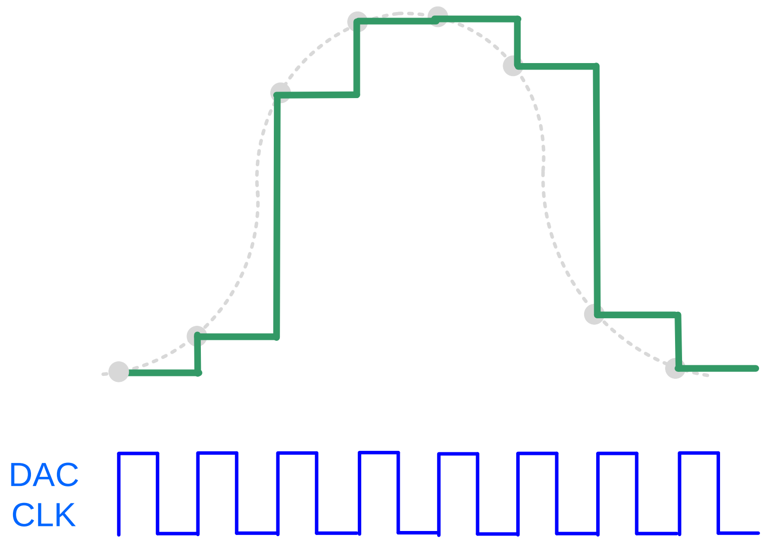


What type of DAC output is desired?

Duty Cycle Effects on Spectral Performance of DACS (File: DAC Quantization with RTZ.m)



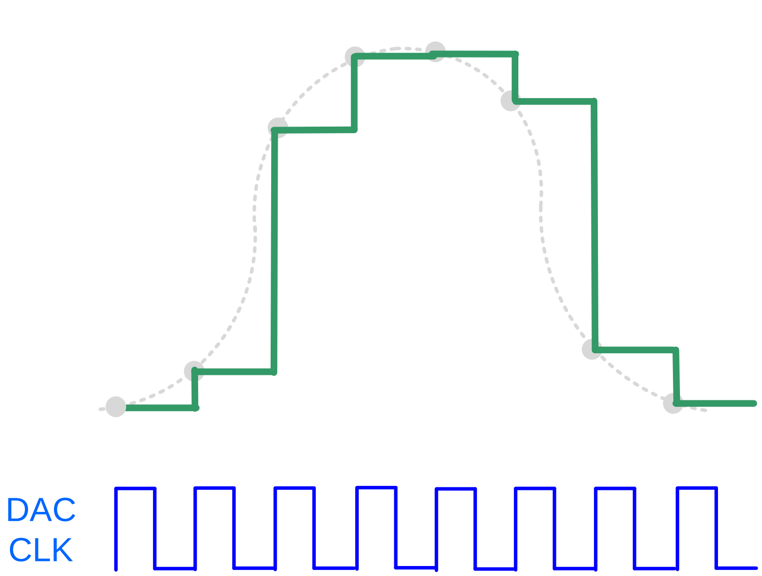
Impulse Output



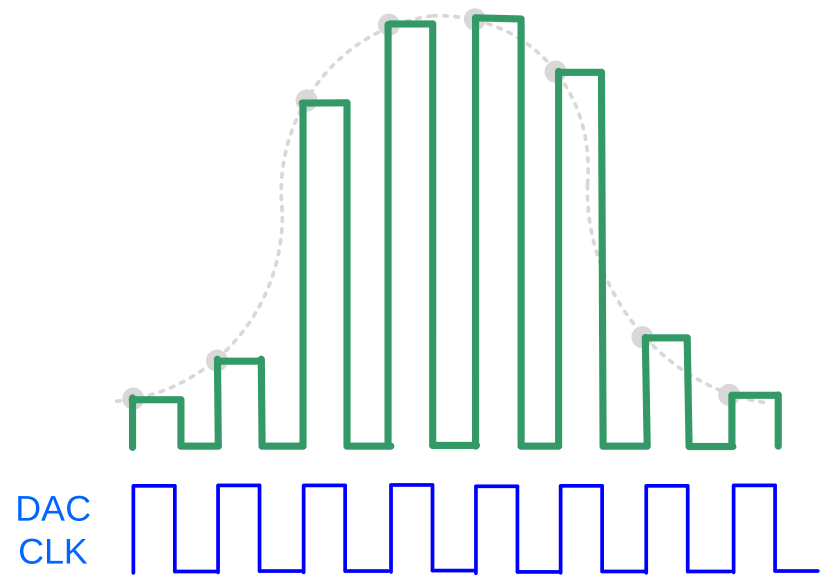
Zero-order Sample and Hold

(100% duty cycle)

Duty Cycle Effects on Spectral Performance of DACS (File: DAC Quantization with RTZ.m)



Zero-order Sample and Hold
(100% duty cycle)



Zero-order Sample and Hold
(50% duty cycle)

Return to Zero

Consider

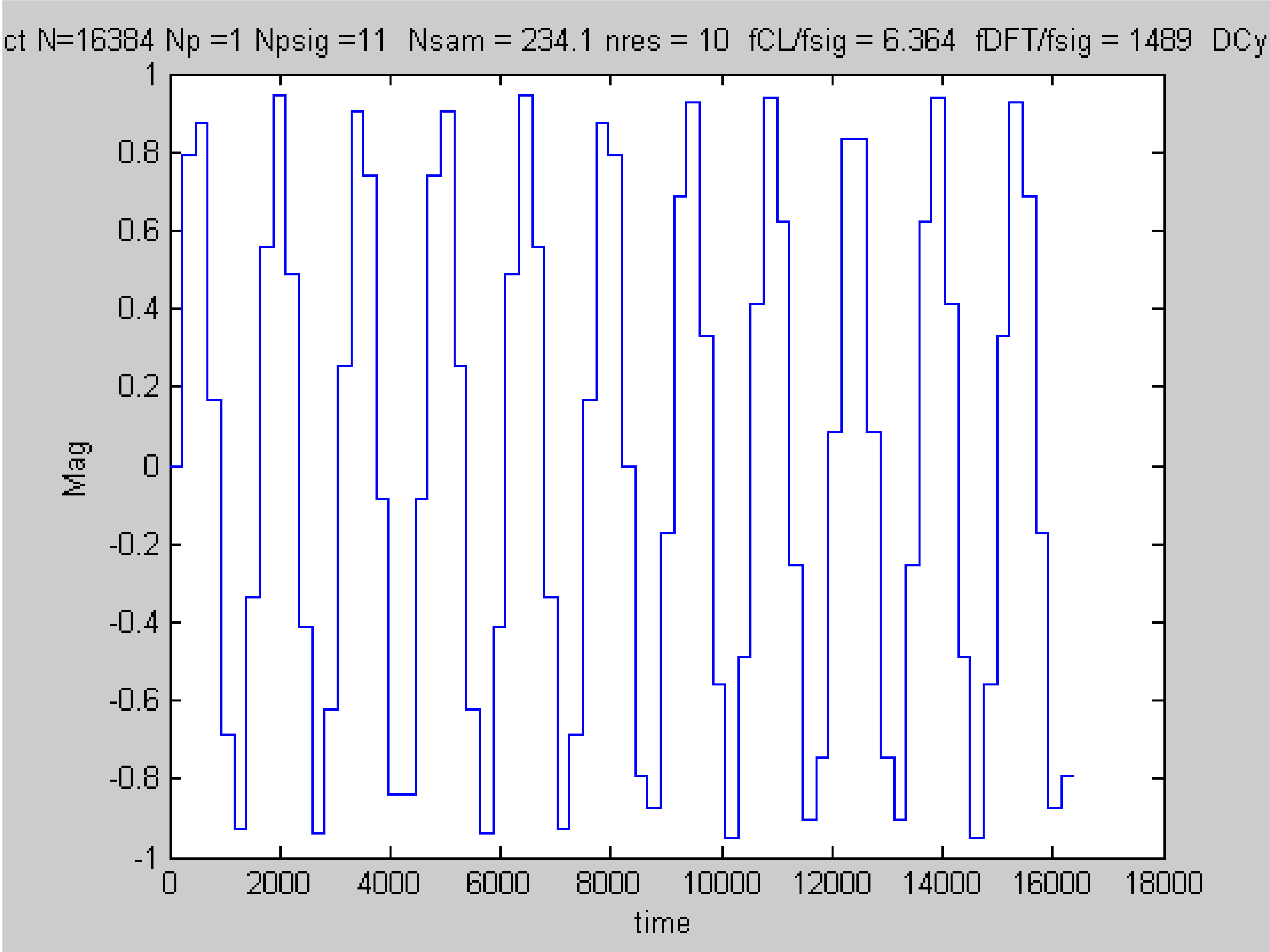
$N_P=1$
$N_{SIG}=11$
$N_{CL}=70$
$f_{sig}=50$
$n_{res}=10$

Thus, $f_{CLK}=f_{SIG}(N_{CL}/N_{SIG})=318Hz$

The fft spectrum should be nominally symmetric around $f_{CLK}/2=159Hz$ so will get only the fundamental, second harmonic, and third harmonic in the fundamental frequency half-period which occurs at fft coefficient number 36 and the clock frequency will be at fft coefficient number 71 (and thus the fundamental will appear at fft coefficient numbers 11+1=12 and 71-11=60)

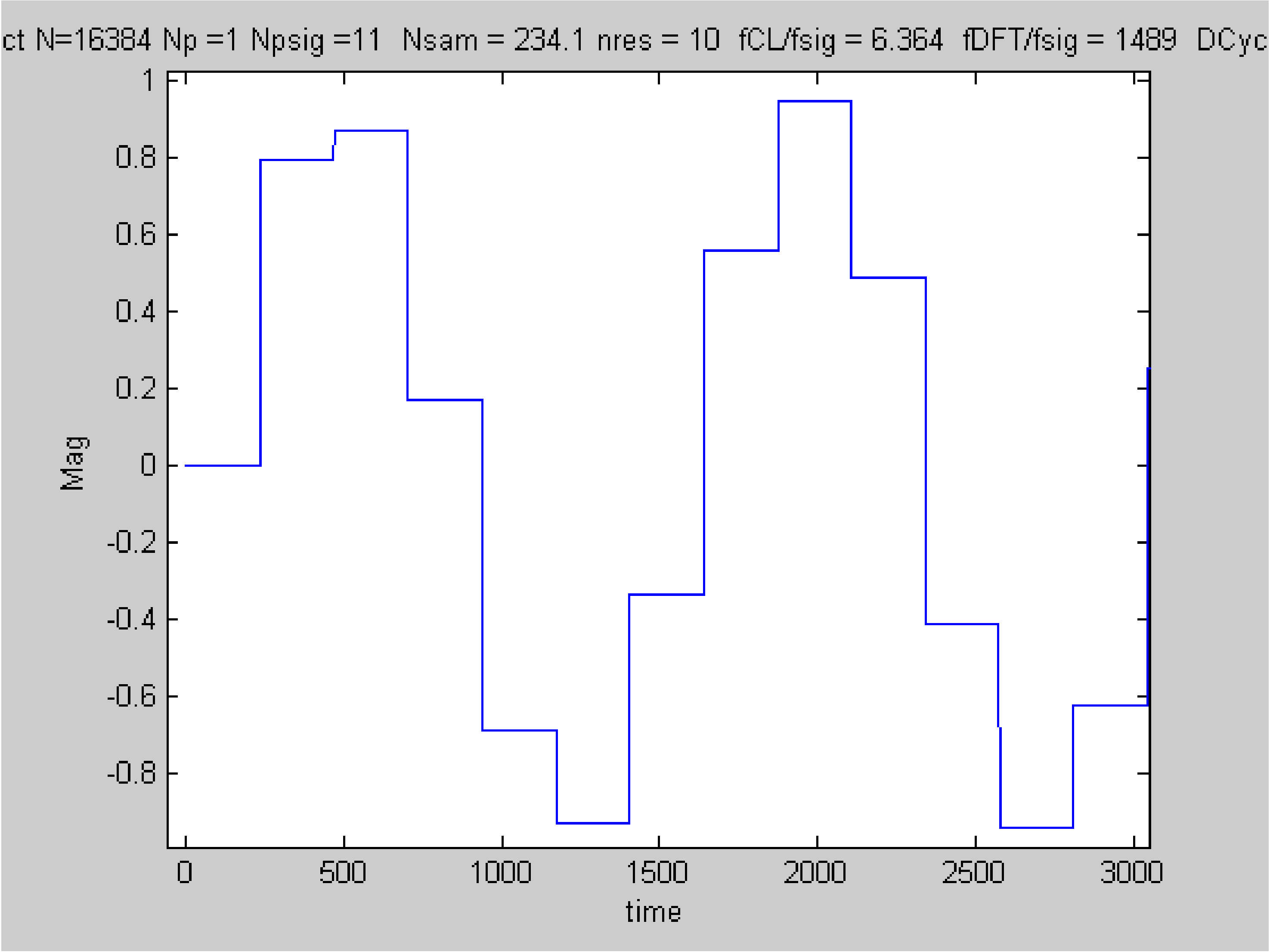
The relationship between fft coefficient number and frequency is given by

$$f=\left(\frac{n-1}{N_{SIG}}\right)f_{SIG} \quad \text{or by} \quad n=1+f\left(\frac{N_{SIG}}{f_{SIG}}\right)$$

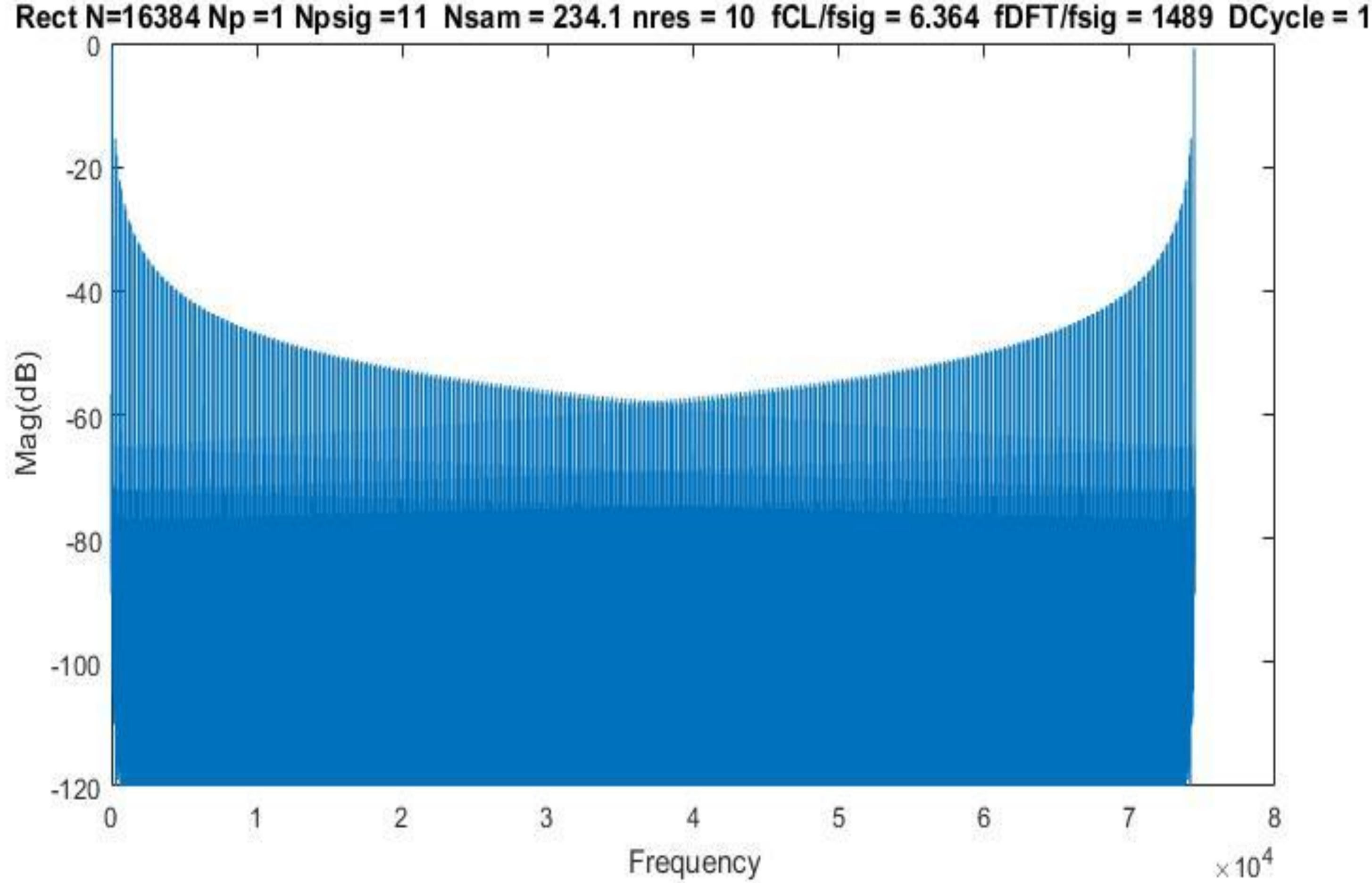


Zero-order Sample and Hold

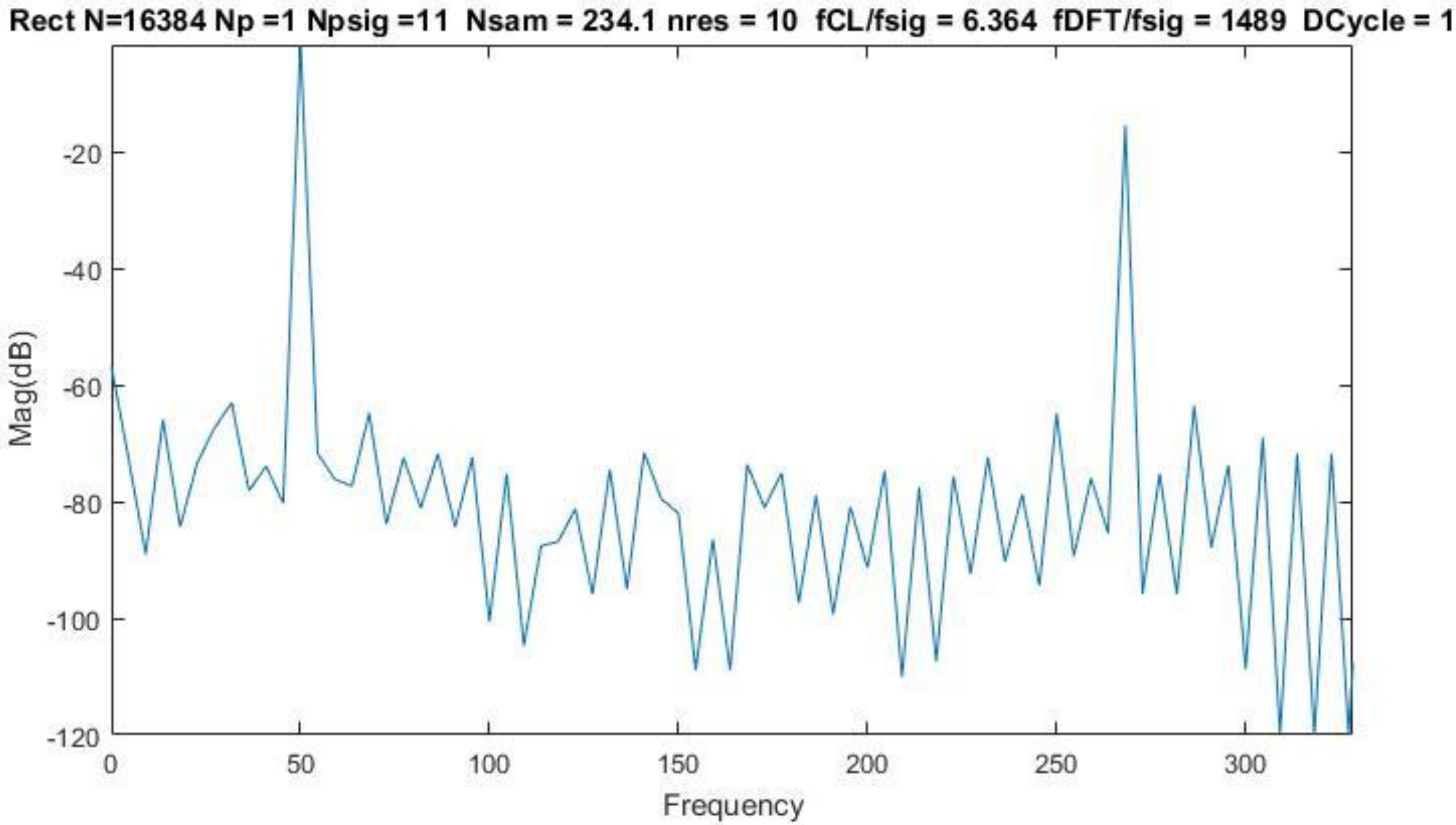
(100% duty cycle)



Zero-order Sample and Hold
(100% duty cycle)



Zero-order Sample and Hold
(100% duty cycle)



Zero-order Sample and Hold
(100% duty cycle)

No spectral distortion components apparent

Magnitude of Fundamental 0.950 2nd Harmonic 0.000
in dB -0.4 -220.0

Res 10 No. points 16384 fsig = 50.00 No.DFT Periods 1.00
No Sig Periods 11.00 fCL/fsig 6.36 Nsamp = 234.06 DutyCycle = 1.0

Rectangular Window
Pyyt =

Columns 1 through 8

-56.7666 -72.6329 -89.0180 -65.9223 -84.2996 -73.2991 -67.3277 -63.0000

Columns 9 through 16

-78.0844 -73.9060 -80.2415 **-0.8009** -71.7226 -76.1473 -77.2781 -64.7624

Columns 17 through 24

-83.8268 -72.4855 -81.0600 -71.7684 -84.3311 -72.4003 **-100.5411** -75.2036

Columns 25 through 32

-104.6890 -87.5996 -86.8260 -81.1797 -95.8796 -74.4617 -94.9546 -71.5626

Columns 33 through 40

-79.4929 -82.0122 -108.8848 -86.4078 -108.8871 -73.6154 -80.9806 -75.0515

Columns 41 through 48

-97.3320 -78.9052 -99.3163 -80.8769 -91.3537 -74.6389 -110.0719 -77.5449

Columns 49 through 56

-107.4100 -75.6450 -92.3523 -72.3248 -90.2704 -78.7130 -94.4099 -64.8687

Columns 57 through 64

-89.3611 -75.9678 -85.3927 -15.3935 -95.8308 -75.1766 -95.9254 -63.5195

Columns 65 through 72

-87.8618 -73.6845 -108.7233 -68.9982 -119.8229 -71.7477 -120.0000 -71.7563

Columns 73 through 80

-119.9494 -68.7360 -109.5559 -73.6204 -89.4074 -63.5185 -97.8093 -74.8683

Columns 81 through 88

-98.2726 -18.1394 -88.4301 -76.0204 -92.7995 -65.1698 -98.4133 -75.7393

Columns 89 through 96

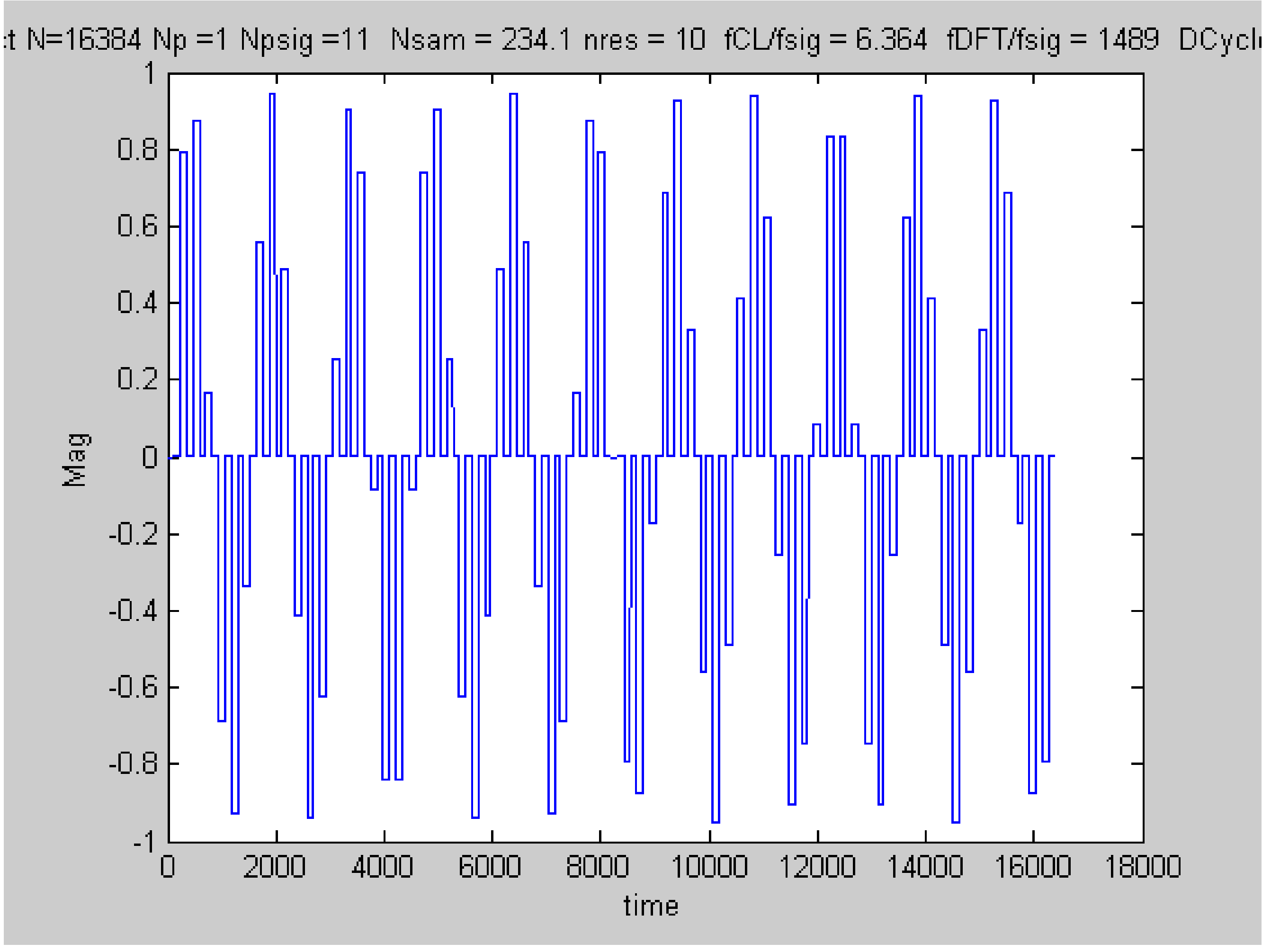
-94.8485 -72.0469 -97.3332 -76.9476 -112.8736 -76.5337 -116.8212 -79.5798

Columns 97 through 104

-98.2141 -81.0207 -106.8397 -76.9870 -105.5319 -79.2621 -89.5668 -79.9400

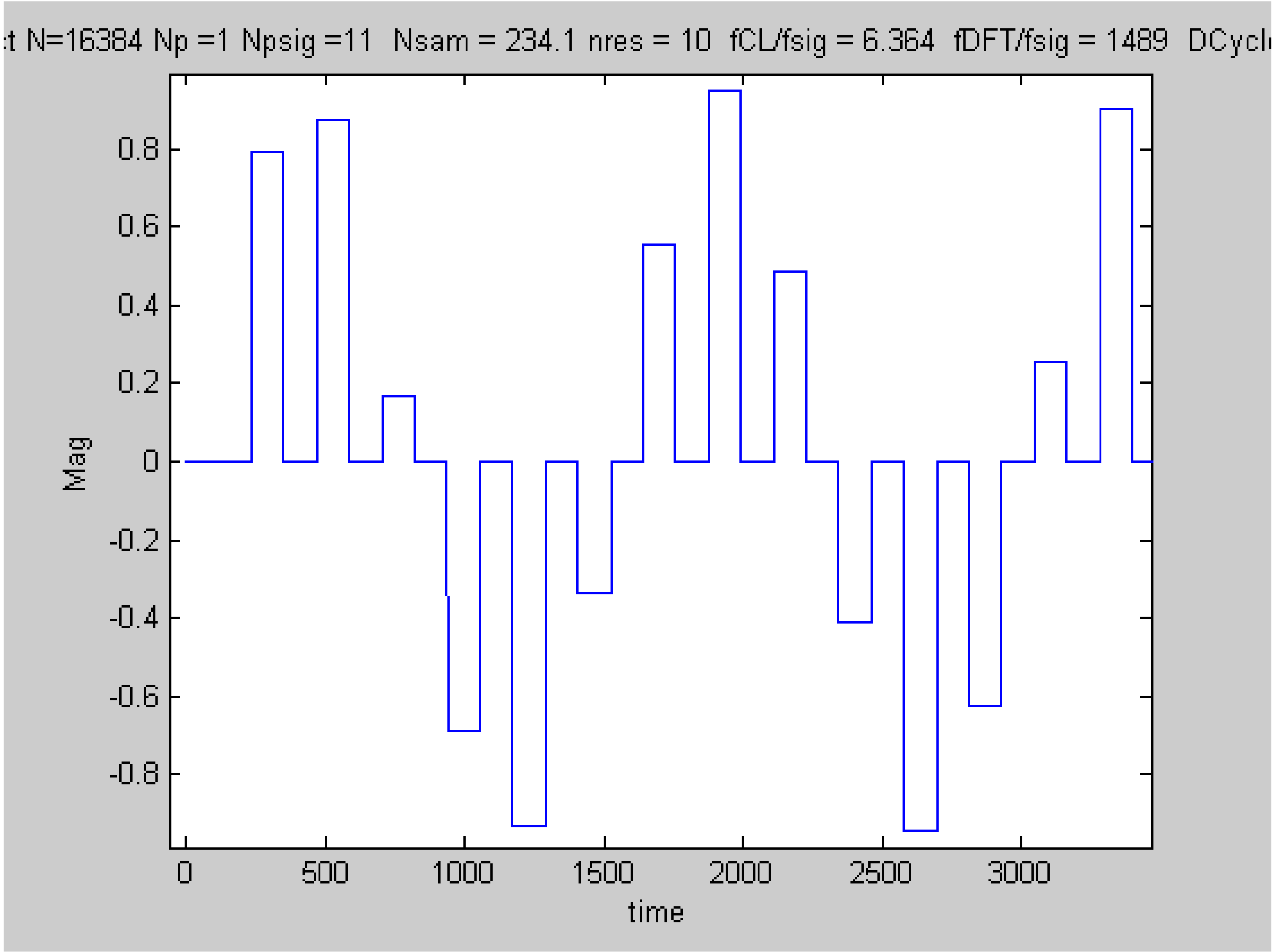
Columns 105 through 110

-118.9287 -86.4077 -117.6606 -76.3449 -90.0484 -82.8245



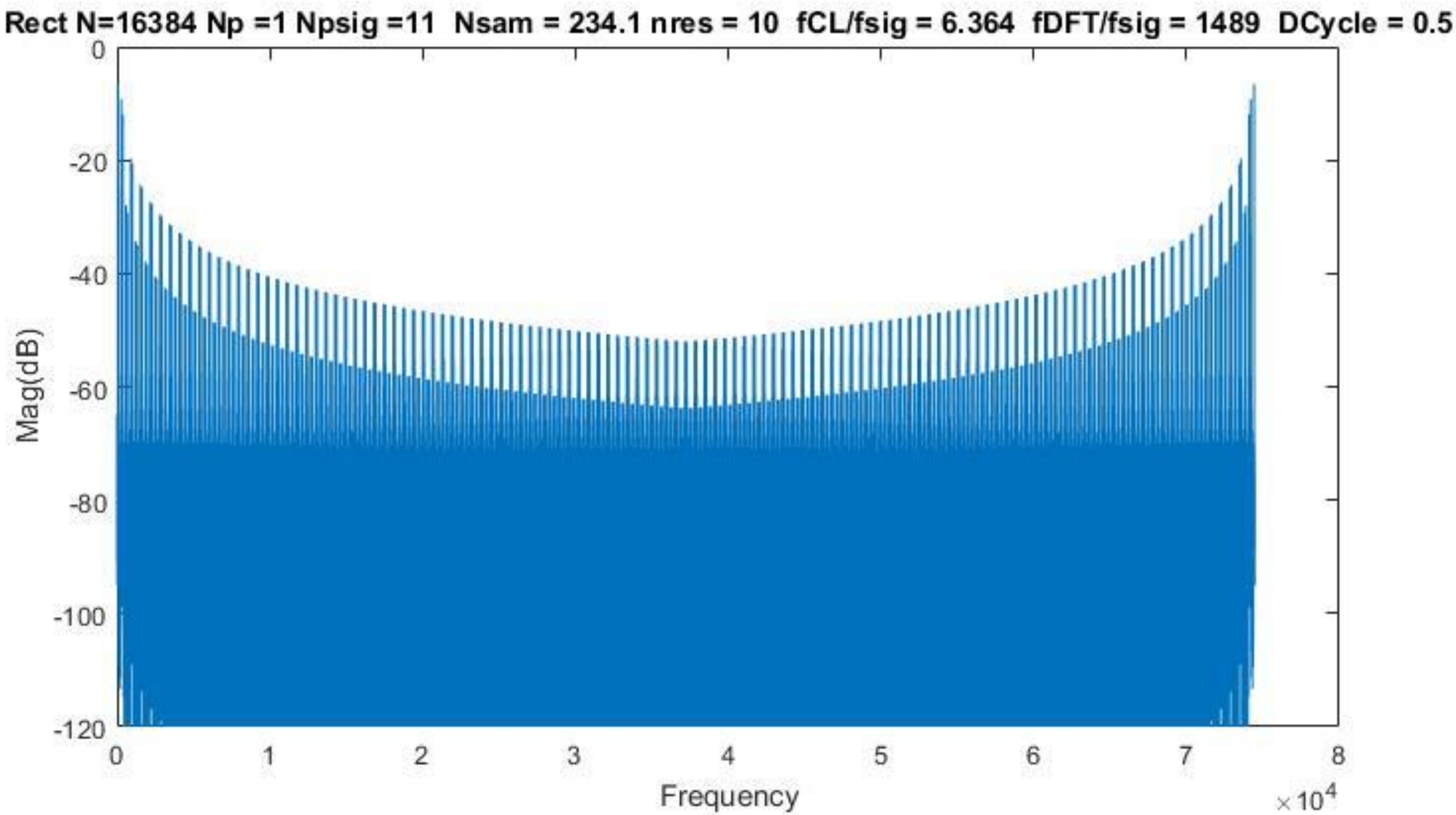
Zero-order Sample and Hold
(50% duty cycle)

Return to Zero

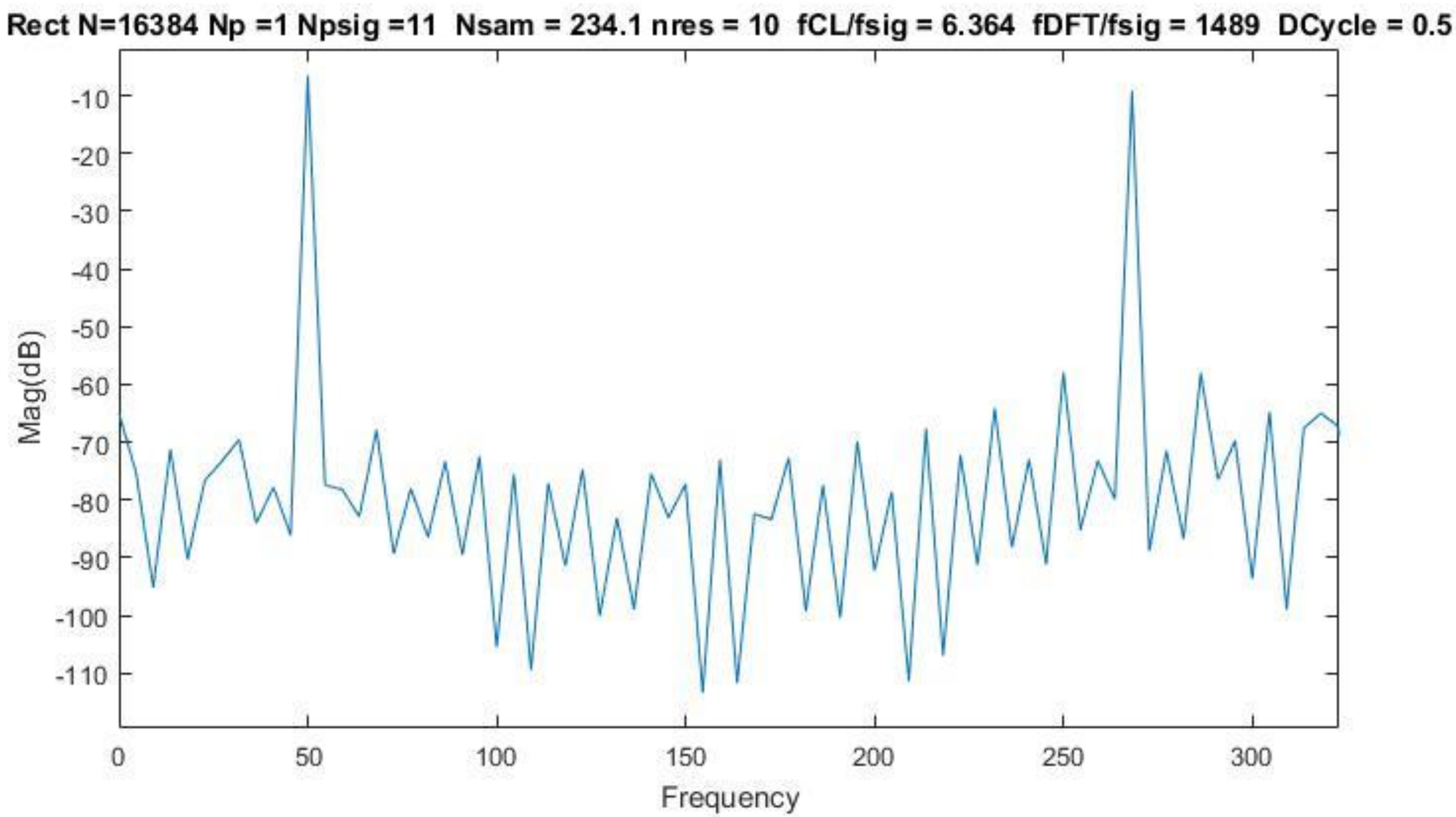


Zero-order Sample and Hold
(50% duty cycle)

Return to Zero



Rect N=16384 Np =1 Npsig =11 Nsam = 234.1 nres = 10 fCL/fsig = 6.364 fDFT/fsig = 1489 DCycle = 0.5
Mag(dB)
0
-20
-40
-60
-80
-100
-120
0
1
2
3
4
5
6
7
8
Frequency
×10^4


Rect N=16384 Np =1 Npsig =11 Nsam = 234.1 nres = 10 fCL/fsig = 6.364 fDFT/fsig = 1489 DCycle = 0.5
Mag(dB)
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
-110
0
50
100
150
200
250
300
Frequency

Magnitude of Fundamental 0.950 2nd Harmonic 0.000
in dB -0.4 -220.0
Res 10 No. points 16384 fsig = 50.00 No.DFT Periods 1.00
No Sig Periods 11.0 fCL/fsig 6.36 Nsamp = 234.06 DutyCycle = 0.5

Rectangular Window

Columns 1 through 8

-64.9875 -75.2613 -95.1326 -71.2094 -90.2852 -76.6156 -73.2632 -69.5014

Columns 9 through 16

-83.9643 -77.8162 -86.0866 **-6.5546** -77.4246 -78.1520 -82.8739 -67.8450

Columns 17 through 24

-89.2754 -77.9987 -86.4061 -73.3492 -89.4464 -72.4374 **-105.4623** -75.4661

Columns 25 through 32

-109.3846 -77.1183 -91.3829 -74.6853 -100.0604 -83.0708 -98.8928 -75.4658

Columns 33 through 40

-83.0515	**-77.2072**	-113.3805	-73.0081	-111.6998	-82.3913	-83.3412	-72.6823

Columns 41 through 48

-99.2516	-77.3944	-100.3706	-69.8376	**-92.1989**	-78.5482	-111.3365	-67.6419

Columns 49 through 56

-106.9480	-72.1848	-91.2497	-64.1067	-88.1852	-72.9575	-91.1348	-57.9429

Columns 57 through 64

-85.1895	-73.1598	-79.8865	-9.1712	-88.8113	-71.4550	-86.7115	-58.0188

Columns 65 through 72

-76.4621	-69.7368	-93.6087	-64.6782	-98.9534	-67.5523	-64.9561	-67.2996

Columns 73 through 80

-98.9315	-63.4010	-94.7247	-69.9035	-77.9584	-58.0242	-89.2150	-71.7656

Columns 81 through 88

-91.2239	-11.9238	-82.9849	-72.9920	-88.6074	-58.0323	-95.6827	-74.9840

Columns 89 through 96

-92.7477	-63.9572	-95.8693	-76.5352	-112.3992	-67.3668	-114.7685	-74.7990

Columns 97 through 104

-99.0492	-69.8650	-109.1443	-75.9390	-107.4431	-70.6650	-92.0134	-75.6831

Columns 105 through 110

-120.0000	-72.9982	-117.8073	-77.0787	-93.6013	-72.8613

DAC Comparisons with Quantization

N	θ	Nsam	n	A_1	A_2	A_3
32K	1	142.5	8	-.596	-56.7	-64.5
128K	1	569.9	8	-.596	-56.7	-64.45
1024	1	6.8	6	-.735	-44.7	-54.1
1024	1	6.8	12	-.594	-80.8	-69.6
1024	1	6.8	24	-.594	-120	-68.5
16K	1	109.2	6	-.729	-44.7	-52.7
16K	1	109.2	12	-.589	-80.8	-90
16K	1	109.2	14	-.589	-120	-92.7
256	1	1.7	18	-.589	-120	-48.2
1024	1	6.8	18	-.595	-120	-68.5
4048	1	27.3	18	-.588	-120	-72.3
16K	1	436.9	18	-.589	-120	-96.5
16K	1	234	10	-.801	-100.5	-82
16K	0.5	234	10	-6.55	-105.4	-77.4

Return to Zero Effects

RTZ reduces signal level

RTZ does not introduce significant distortion

RTZ typically degrades SNR

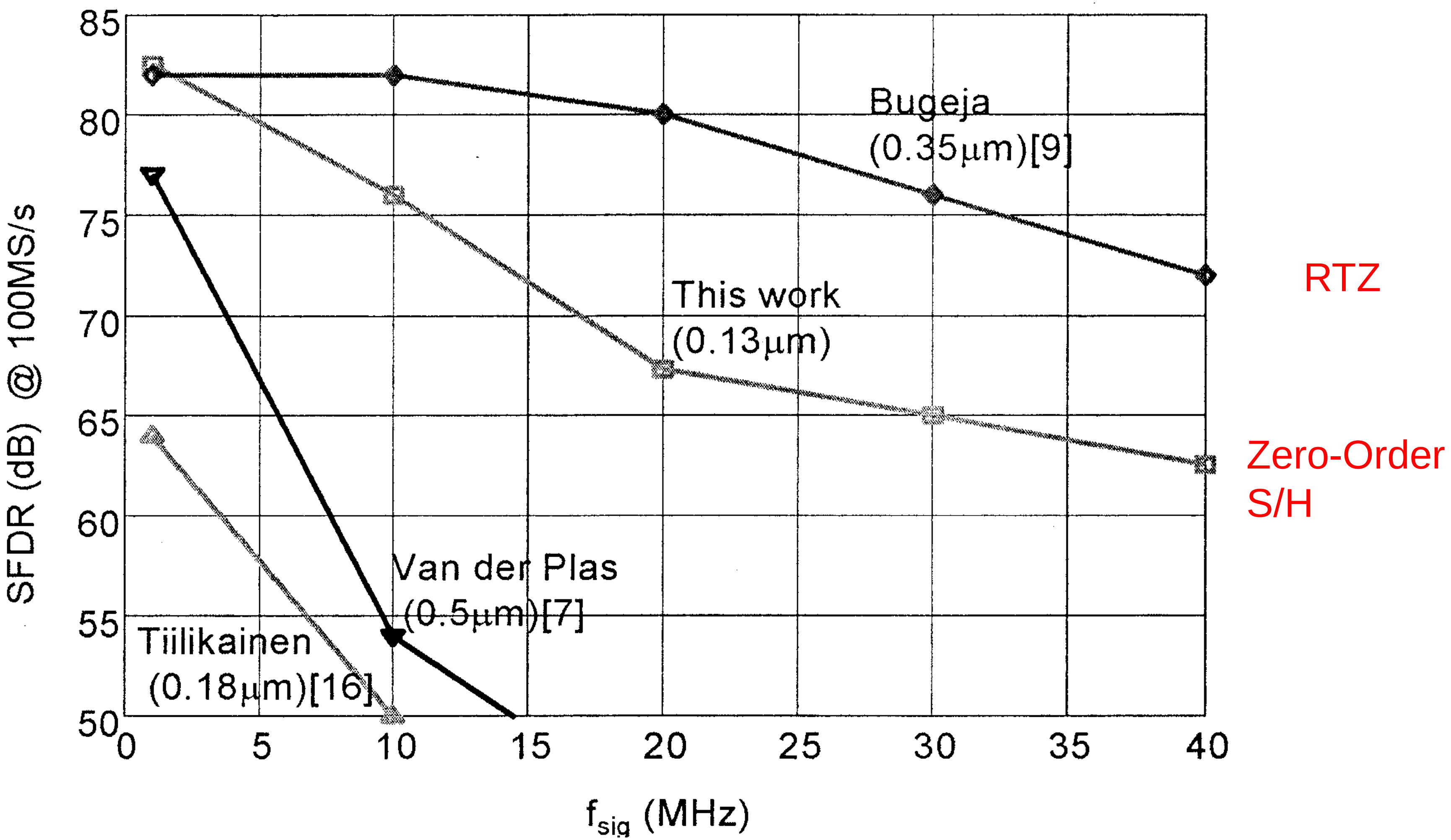
Previous-code dependence in a data converter can introduce significant distortion and this is often significant when operating with high-frequency inputs and high-speed clocks

Return-to Zero can significantly reduce previous-code dependence

RTZ may significantly improve SDR (or SFDR or THD)

Effects of RTZ on SNDR are less apparent since SDR improves but SNR deteriorates but in a good design, the distortion improvements with RTZ may be sufficiently attractive to overcome the loss in SNR

Typical SFDR Plots



From: Y. Cong and R. L. Geiger, "A 1.5-v 14-bit 100-MS/s Self-Calibrated DAC," *IEEE J. of Solid State Circuits*, December 2003, vol. 38, no. 12, pp. 2051-2060.

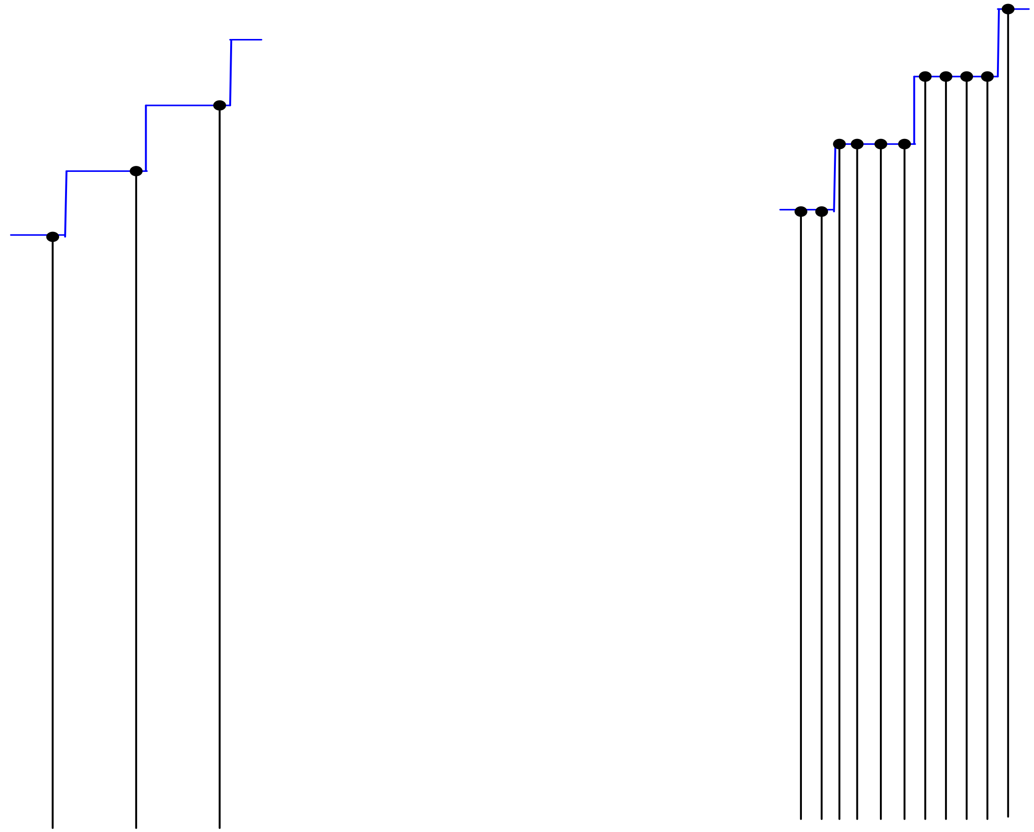
Summary of Duty Cycle Effects

Duty Cycle does not introduce harmonic distortion

Duty Cycle reduction reduces signal levels thus degrades SNR

Duty Cycle reduction to achieve RTZ can improve SDR and SNDR

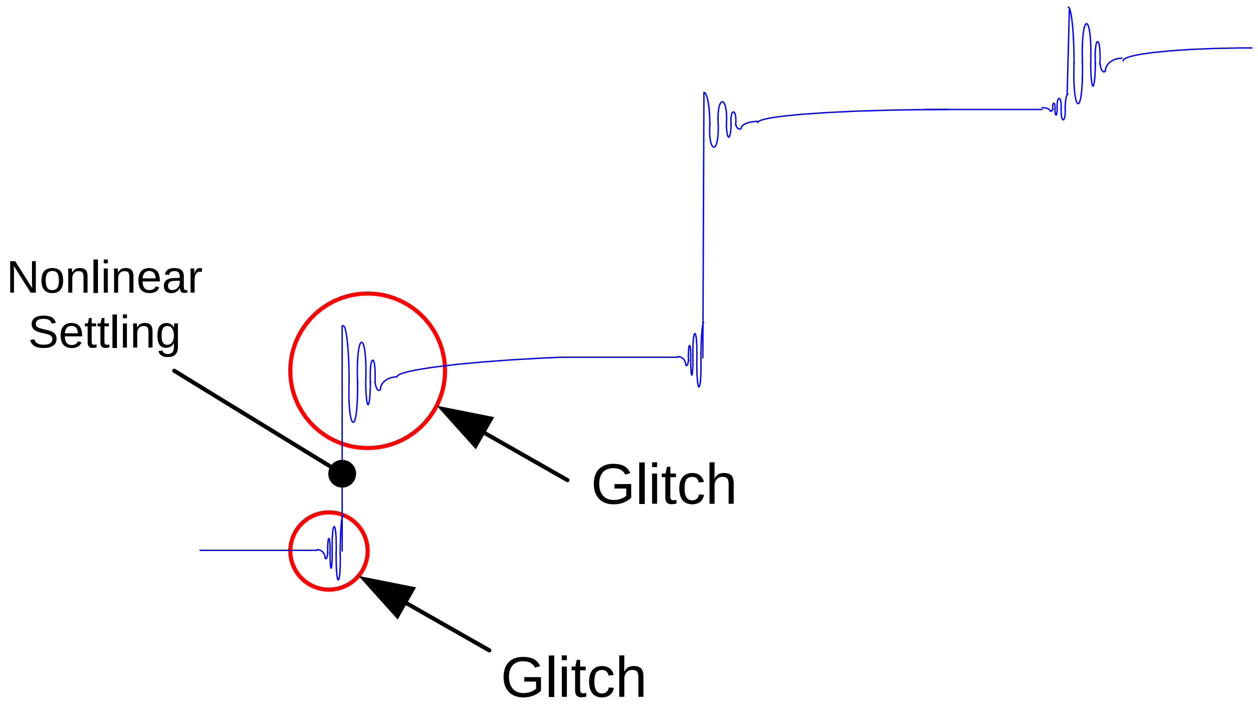
Number of Samples/Period



One Sample per Period | Multiple Samples per Period

- Many authors use a data acquisition system and select one sample/period
- Spectrum analyzer will generally measure continuous-time effects
- What is most important in the DAC output is strongly system application dependent

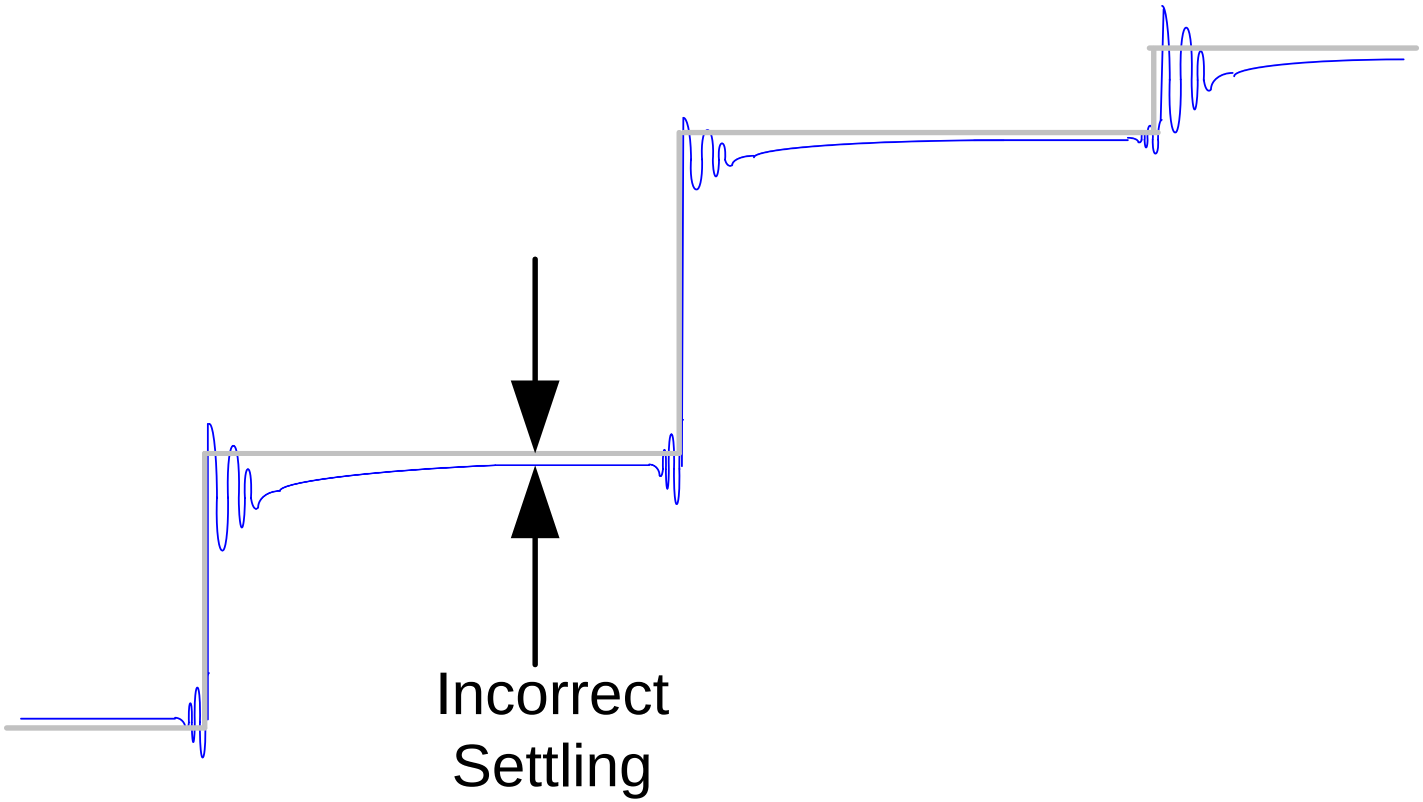
Number of Samples/Period



Typical DAC Response

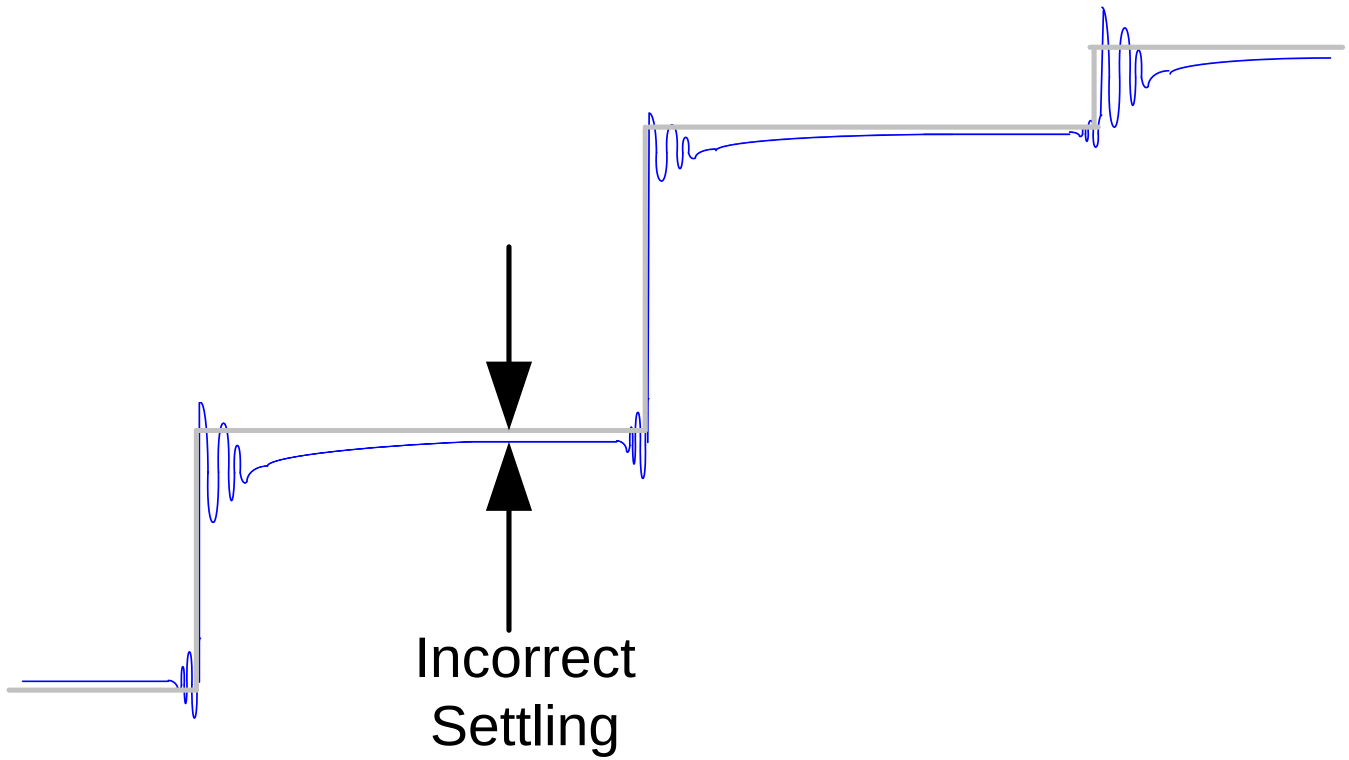
Glitches for even small changes in DAC output for some architectures can be very large (hundreds or even thousands of LSBs)

Number of Samples/Period



Typical DAC Response

Number of Samples/Period



- Settling error can be multiple LSB at Nyquist Rate
- Multiple LSB settling error does not cause distortion if settling is linear
- Glitches are a significant contributor to spectral distortion (at high frequencies)

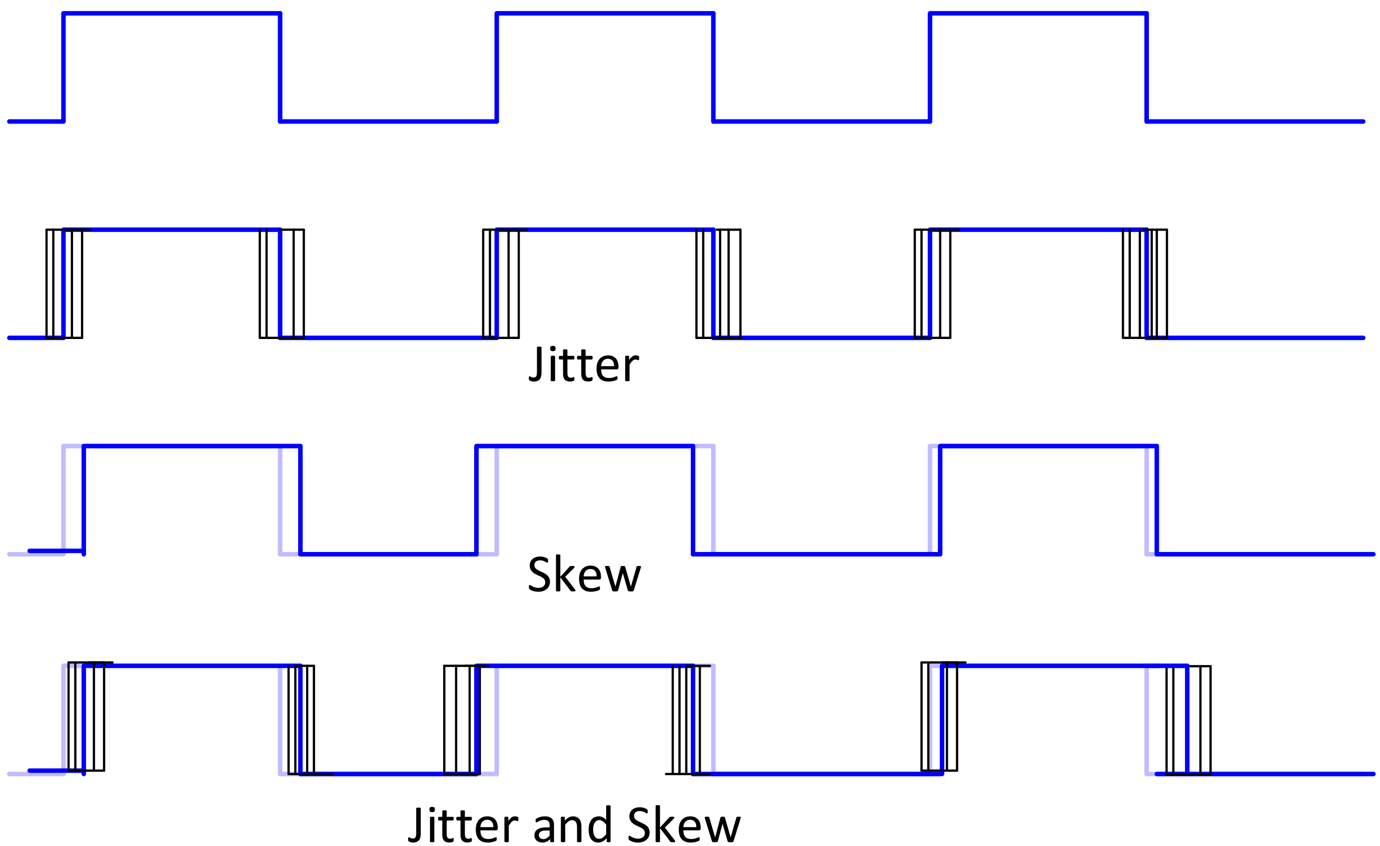
Spectral Characterization of Data Converters

- Distortion Analysis
- Time Quantization Effects
  - of DACs
  - of ADCs
- Amplitude Quantization Effects
  - of DACs
  - of ADCs

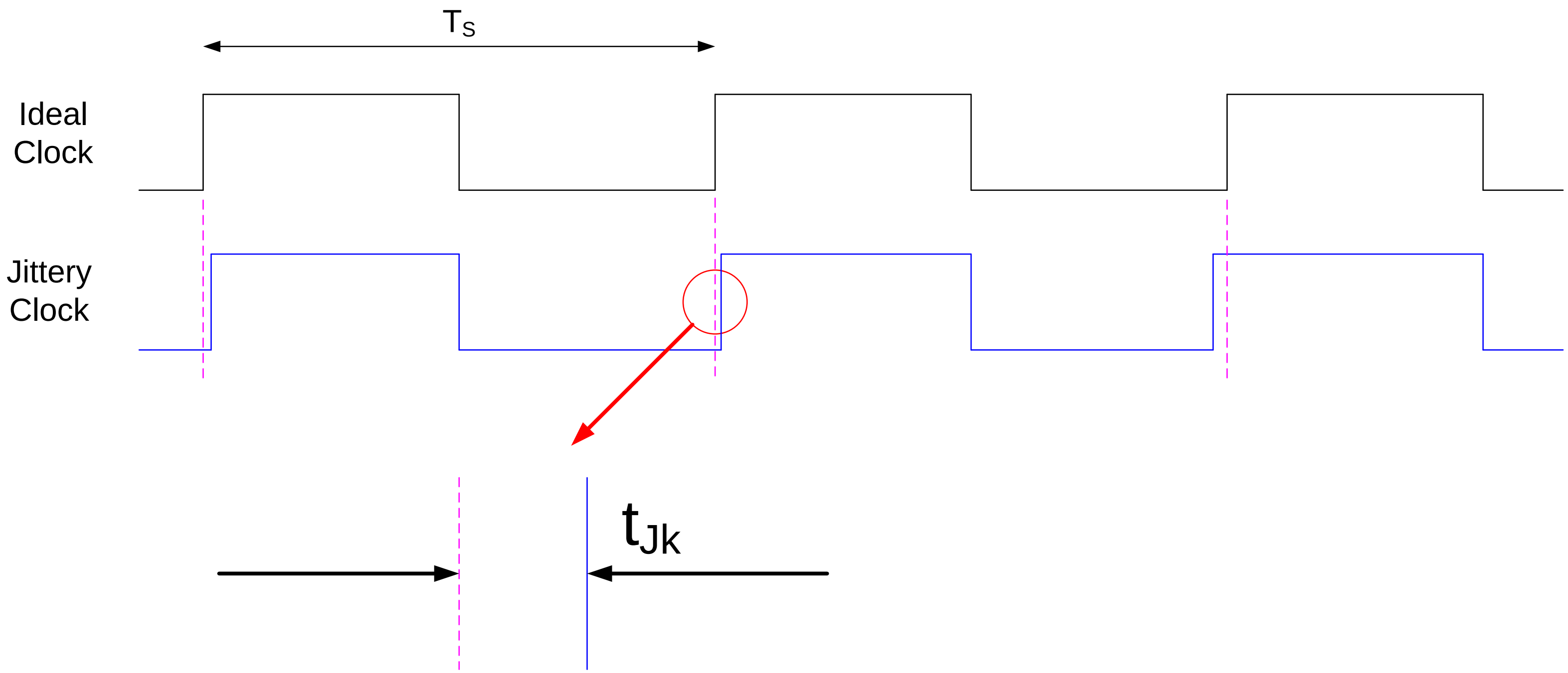
Clock Jitter

Effects of Jitter on Spectral Performance

Jitter and Skew



Model of Jitter



Assume t_{Jk} are uncorrelated uniformly distributed random variables

$$t_{Jk} \propto U\left(-\frac{\theta}{2}T_S, \frac{\theta}{2}T_S\right)$$

Note: there can also be jitter in the ideal clock or there may be no ideal clock so zero crossings may be modeled as a random walk or a sum of a random walk and uniform jitter. Analysis more complicated in these cases.

Analytical Characterization of Clock Jitter

Assume the input can be expressed as

$$v_{IN} = \frac{V_{REF}}{2} + \frac{V_{REF}}{2}\sin(\omega t + \theta)$$

Rather than assuming that the clock has jitter and the input has no jitter, it will be assumed that the clock has no jitter but the input contains the jitter. This should provide the same jitter-based sampling errors. Thus, it will be assume that the time variable in the input can be expressed as

$$t = t_N + t_R$$

where t_N the nominal time and t_R is the random time (that has been added to the input rather than the clock)

The input can be expanded in a Taylor's series as

$$v_{IN} = v_{IN}\Big|_{t_R=0} + \frac{\partial v_{IN}}{\partial t_R}\Big|_{t_R=0} t_R + \frac{1}{2!}\frac{\partial^2 v_{IN}}{\partial t_R^2}\Big|_{t_R=0} t_R^2 + \;....$$

Truncating after first-order terms we have

$$v_{IN} \cong v_{IN}\Big|_{t_R=0} + \frac{\partial v_{IN}}{\partial t_R}\Big|_{t_R=0} t_R$$

Analytical Characterization of Clock Jitter

$$v_{\text{IN}} \cong v_{\text{IN}}\Big|_{t_R=0} + \frac{\partial v_{\text{IN}}}{\partial t_R}\Big|_{t_R=0} t_R$$

It now follows from the expression from the input that

$$\frac{\partial v_{\text{IN}}}{\partial t_R}\Big|_{t_R=0} = \frac{V_{REF}}{2}\omega\cos(\omega t_N + \theta)$$

Thus

$$v_{\text{IN}} \cong \frac{V_{REF}}{2} + \frac{V_{REF}}{2}\sin(\omega \text{t}_N + \theta) + \frac{V_{REF}}{2}\omega\cos(\omega(t_N) + \theta)t_R$$

The signal and noise jitter components can be identified as

$$v_{\text{IN\_Sig}} \cong \frac{V_{REF}}{2} + \frac{V_{REF}}{2}\sin(\omega \text{t}_N + \theta)$$

$$v_{\text{IN\_jitter}} \cong \frac{V_{REF}}{2}\omega\cos(\omega(t_N) + \theta)t_R$$

Analytical Characterization of Clock Jitter

$$v_{\text{IN\_Sig}} \cong \frac{V_{REF}}{2} + \frac{V_{REF}}{2}\sin\left(\omega t_N + \theta\right)$$

$$v_{\text{IN\_jitter}} \cong \frac{V_{REF}}{2}\omega\cos\left(\omega\left(t_N\right)+\theta\right)t_R$$

Will now obtain the $\text{SNR}_{\text{Jitter}}$

Observe the jitter noise can be expressed as

$$v_{\text{IN\_jitter}} \cong \left[\frac{V_{REF}}{2}\omega\cos\left(\omega\left(t_N\right)+\theta\right)\right]\bullet t_R$$

Consider the following theorem:

Theorem: If $X_1(t)$ is a zero-mean random process and $X_2(t)$ is a periodic deterministic function where the RMS value of X_1 is X_{1RMS} and the RMS value of X_2 is X_{2RMS}, then the RMS value of the product is given by the expression $X_{RMS}=X_{1RMS}X_{2RMS}$

Analytical Characterization of Clock Jitter

$$v_{\text{IN\_jitterRMS}} \cong \left[\frac{V_{REF}}{2}\omega\cos\left(\omega\left(t_N\right)+\theta\right)\right]\Bigg|_{RMS} \bullet t_R\Big|_{RMS}$$

$$\left[\frac{V_{REF}}{2}\omega\cos\left(\omega\left(t_N\right)+\theta\right)\right]\Bigg|_{RMS} = \left[\frac{V_{REF}}{2}\frac{\omega}{\sqrt{2}}\right]$$

Recall it has been assumed that at the zero crossings of the sampling clock

$$t_R \propto U\left(-\frac{\theta}{2}T_S, \frac{\theta}{2}T_S\right) \qquad \mu_{t_R}=0 \qquad \sigma_{t_R}=\frac{\theta T_S}{\sqrt{12}}$$

Recall another theorem

Theorem: If n(t) is a random process and <$n(kT_S)$> is a sequence of samples of n(t) then for large T/T_S,

$$V_{RMS}=\sqrt{\frac{1}{T}\int_{t_1}^{t_1+T} n^2(t)dt} = \sqrt{\sigma^2_{n(kT_S)}+\mu^2_{n(kT_S)}}$$

Thus the RMS value of the jitter time sequence obtained by sampling the jitter at multiples of the nominal sampling period T can be expressed as

$$t_R\Big|_{RMS} = \sigma_{t_R} = \frac{\theta T_S}{\sqrt{12}}$$

Analytical Characterization of Clock Jitter

$$v_{\text{IN\_jitterRMS}} \cong \left[\frac{V_{REF}}{2}\omega\cos\left(\omega\left(t_N\right)+\theta\right)\right]\Bigg|_{RMS} \bullet t_R\Big|_{RMS}$$

We thus have

$$v_{\text{IN\_jitterRMS}} \cong \left[\frac{\frac{V_{REF}}{2}\omega}{\sqrt{2}}\right] \bullet \sigma_{t_R}$$

For full-signal input, the RMS value is given by

$$v_{\text{IN\_SigRMS}} \cong \frac{V_{REF}}{2\sqrt{2}}$$

It thus follows that the SNR is given by

$$SNR_{Jitter} = \frac{\frac{V_{REF}}{2\sqrt{2}}}{\left[\frac{\frac{V_{REF}}{2}\omega}{\sqrt{2}}\right] \bullet \sigma_{t_R}} = \frac{1}{\omega\sigma_{t_R}}$$

Analytical Characterization of Clock Jitter

$$SNR_{Jitter} = \frac{1}{\omega\sigma_{t_R}}$$

Or in dB we thus have

$$SNR_{Jitter\_dB} = -20\log\left(2\pi f\sigma_{t_R}\right)$$

$$SNR_{Jitter\_dB} = -15.96 - 20\log\left(f\sigma_{t_R}\right)$$

For small f or σ_{tR} the right-most term is large and positive

This can be compared to the quantization noise

$$SNR_{Quant\_dB} = 6.02n + 1.76$$

As the f σ_{tR} product gets large, the jitter will dramatically degrade performance

Combined Quantization and Jitter Noise

$$v_{\text{noiseRMS}} = \sqrt{v_{QuantRMS}^2 + v_{\text{IN\_jitterRMS}}^2}$$

Recall

$$v_{Quant\text{RMS}} = \frac{V_{LSB}}{\sqrt{12}} = \frac{V_{REF}}{2^n\sqrt{12}}$$

$$v_{Sig\text{RMS}} = \frac{V_{REF}}{2\sqrt{2}}$$

Thus

$$SNR_{Jitter-Quant} = \frac{\frac{V_{REF}}{2\sqrt{2}}}{\sqrt{\left[\frac{\frac{V_{REF}}{2}\omega}{\sqrt{2}}\right]^2 \sigma_{t_R}^2 + \frac{V_{REF}^2}{2^{2n} \bullet 12}}} = \frac{1}{\sqrt{\omega^2\sigma_{t_R}^2 + \frac{8}{3 \bullet 2^{2n+2}}}}$$

Alternately

$$SNR_{Jitter-Quant} = \frac{1}{\sqrt{\frac{1}{SNR_{Jitter}^2} + \frac{1}{SNR_{Quant}^2}}}$$

$$SNR_{Jitter-QuantdB} = -10\log\left(\frac{1}{SNR_{Jitter}^2} + \frac{1}{SNR_{Quant}^2}\right)$$

Combined Quantization and Jitter Noise

$$SNR_{Jitter-Quant} = \frac{1}{\sqrt{\omega^2 \sigma_{t_R}^2 + \frac{8}{3 \bullet 2^{2n+2}}}}$$

Crossover Frequency

$$f = \frac{1}{\pi \sigma_{t_R}} \sqrt{\frac{8}{3}} \frac{1}{2^{n+2}} = \frac{0.13}{\sigma_{t_R} 2^n}$$

Model of Jitter

Assume t_{Jk} are uncorrelated uniformly distributed random variables

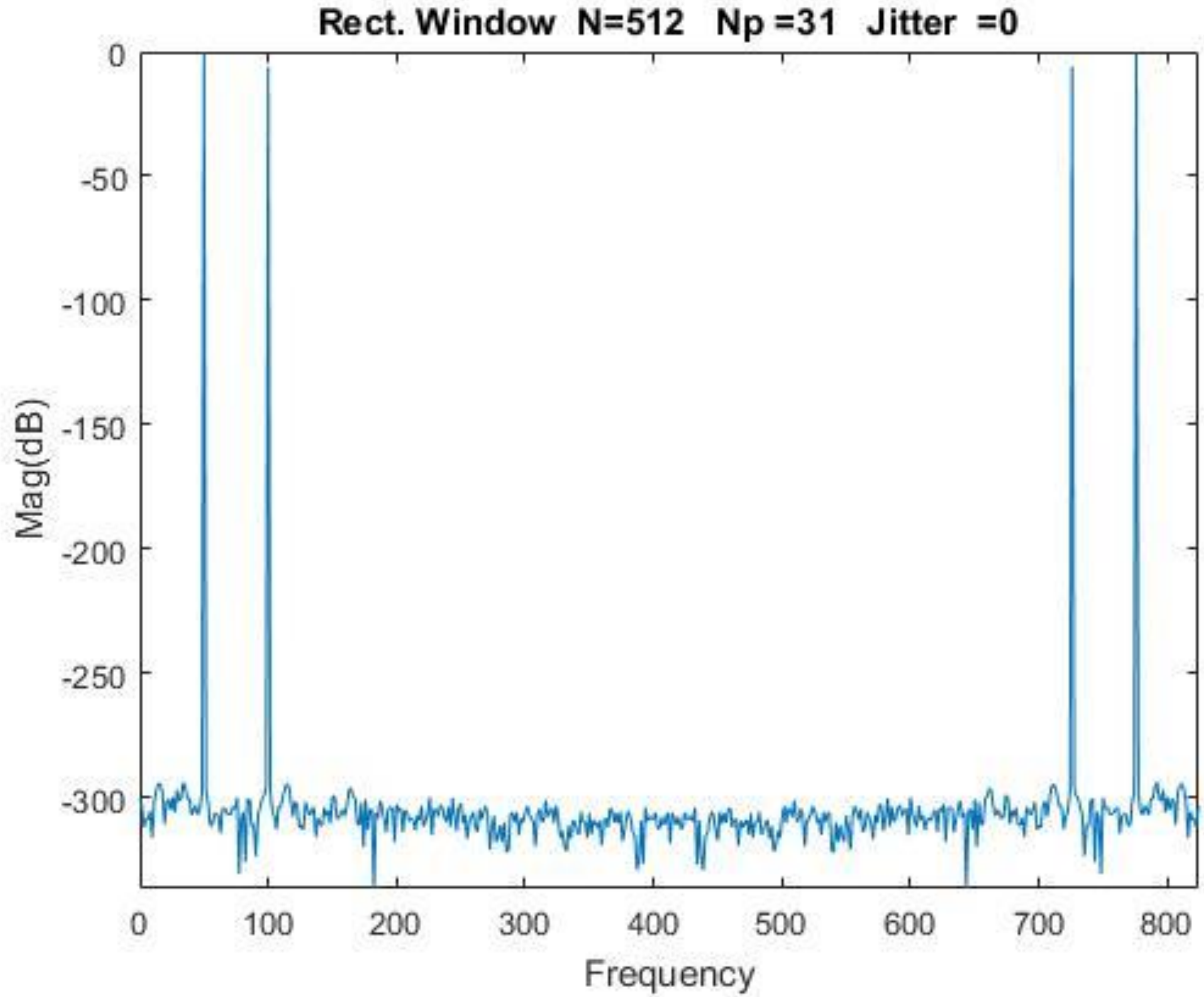
$$t_{Jk} \propto U\left(-\frac{\theta}{2}T_S,\ \frac{\theta}{2}T_S\right)$$

Consider θ=.01, .001, .0001, .00001

Observe: If T_S is a 100MHz clock, then T_S=10nsec and θ=.0001 corresponds to 1psec (±0.5psec) of symmetric jitter

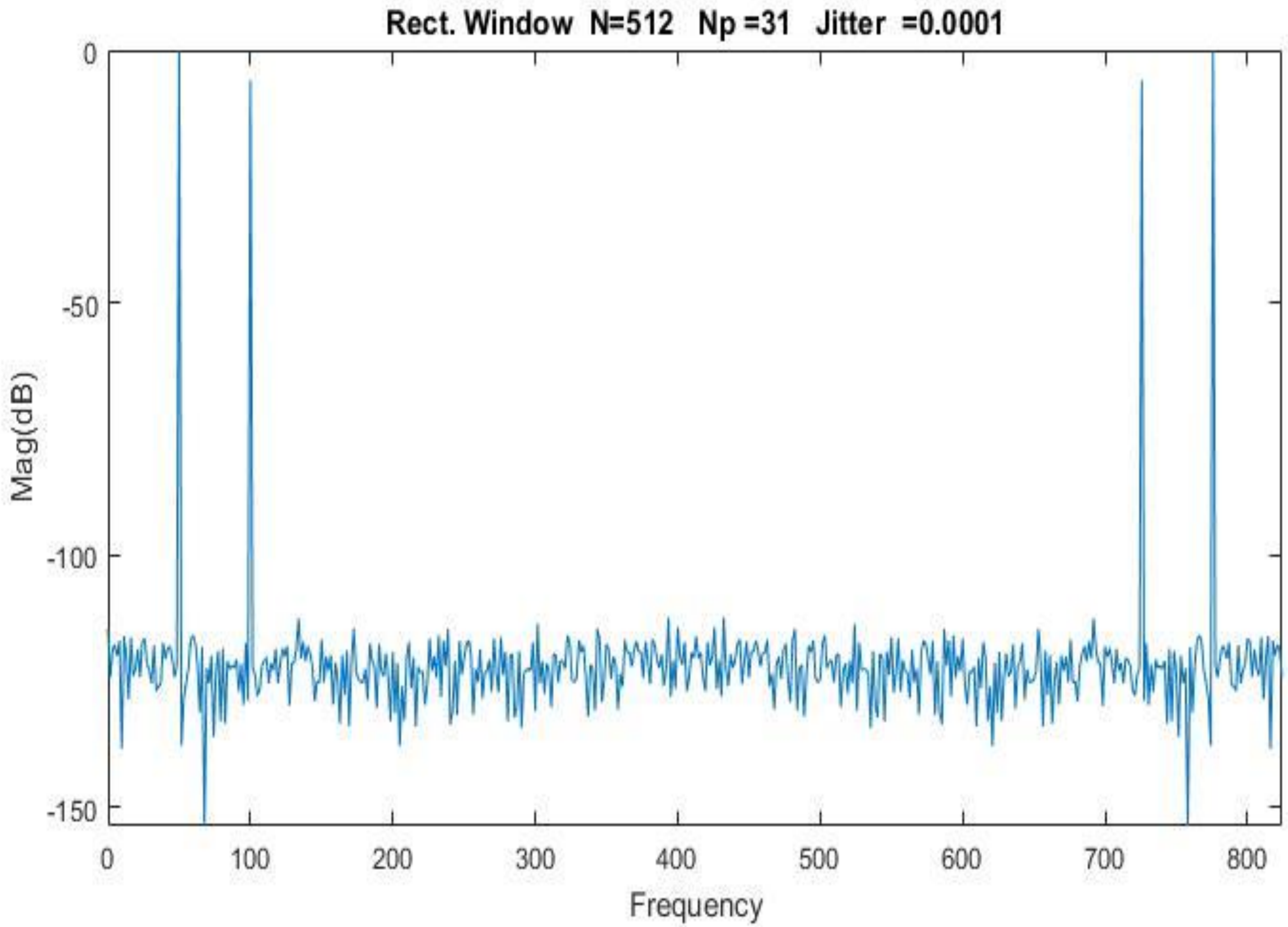
Effects of jitter on spectral performance

$$V_{IN} = \sin\left(\omega t\right) + 0.5\sin(2\omega t) \qquad \omega = 2\pi f_{sig} \qquad f_{sig} = 50\text{Hz}$$



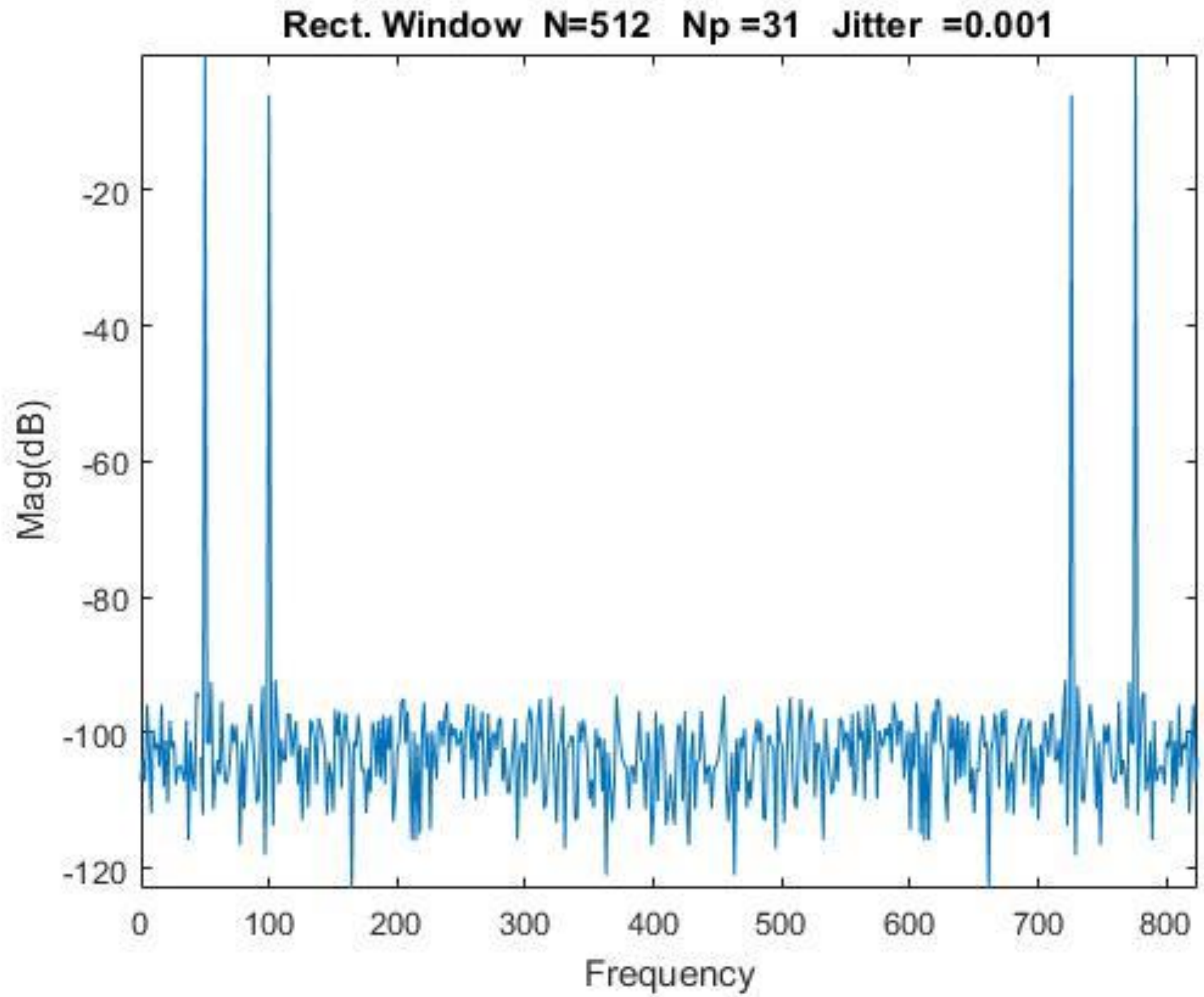
Effects of jitter on spectral performance

$$V_{IN} = \sin(\omega t) + 0.5\sin(2\omega t) \qquad \omega = 2\pi f_{sig} \qquad f_{sig} = 50\text{Hz}$$



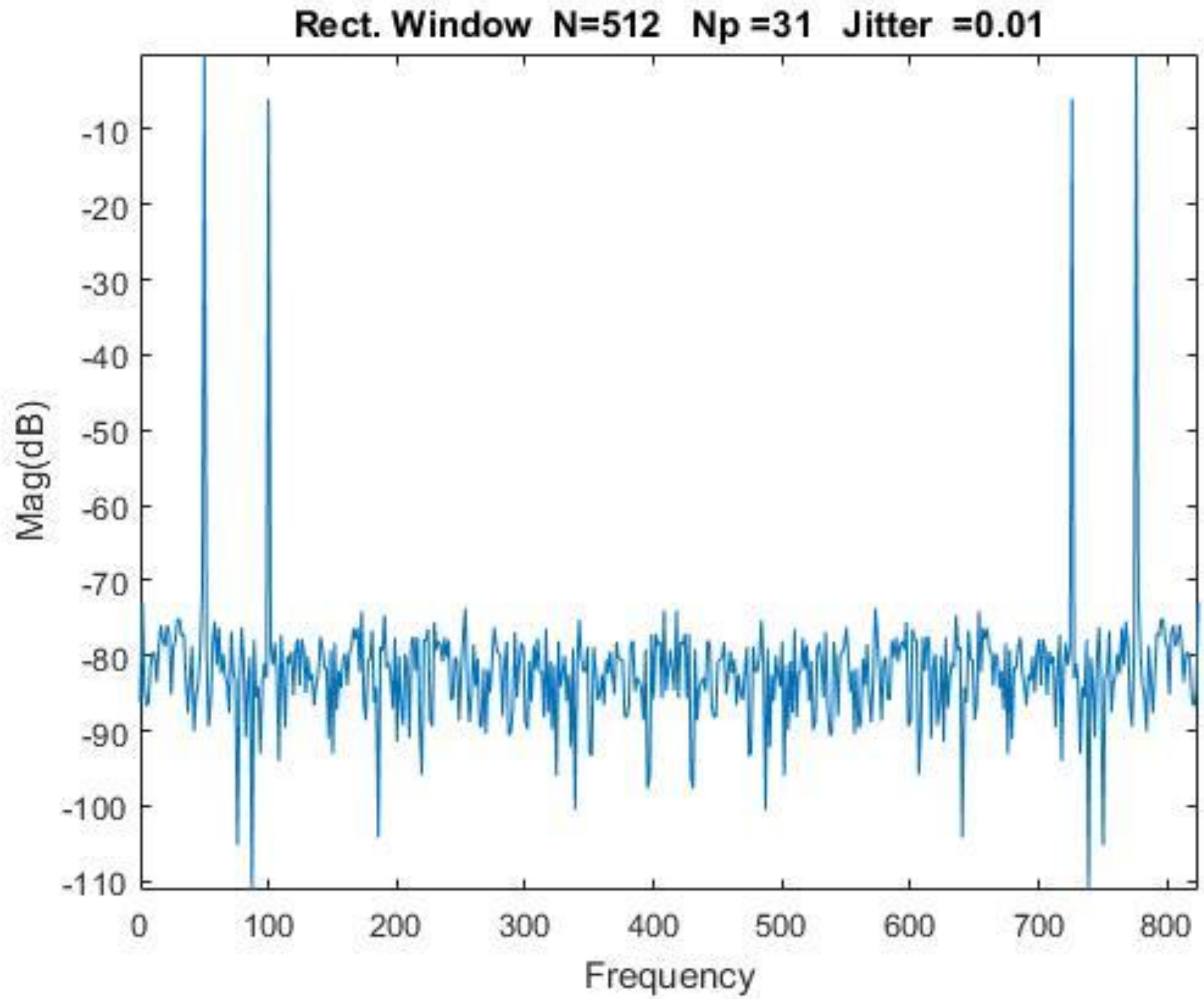
Effects of jitter on spectral performance

$$V_{IN} = \sin(\omega t) + 0.5\sin(2\omega t) \qquad \omega = 2\pi f_{sig} \qquad f_{sig} = 50\text{Hz}$$



Effects of jitter on spectral performance

$$V_{IN} = \sin(\omega t) + 0.5\sin(2\omega t) \qquad \omega = 2\pi f_{sig} \qquad f_{sig} = 50\text{Hz}$$



Summary of Jitter Effects

Jitter (as considered here) does not introduce harmonic distortion

Jitter does increase the noise floor

Jitter vs Clock Skew

- Jitter and Clock skew may appear to be closely related but have dramatically different effects
- Clock Skew is a systematic perturbation of the clock signal
- Clock Skew may be a random variable at the design stage but each fabricated device will have a specific clock skew
- Clock edge variations from ideal will be the sum of those variations due to random noise and those due to clock skew
- In contrast to jitter which does not introduce harmonic distortion, clock skew can introduce spectral components, specifically harmonic components and spectral spreading around the spectral components of the fundamental and harmonics

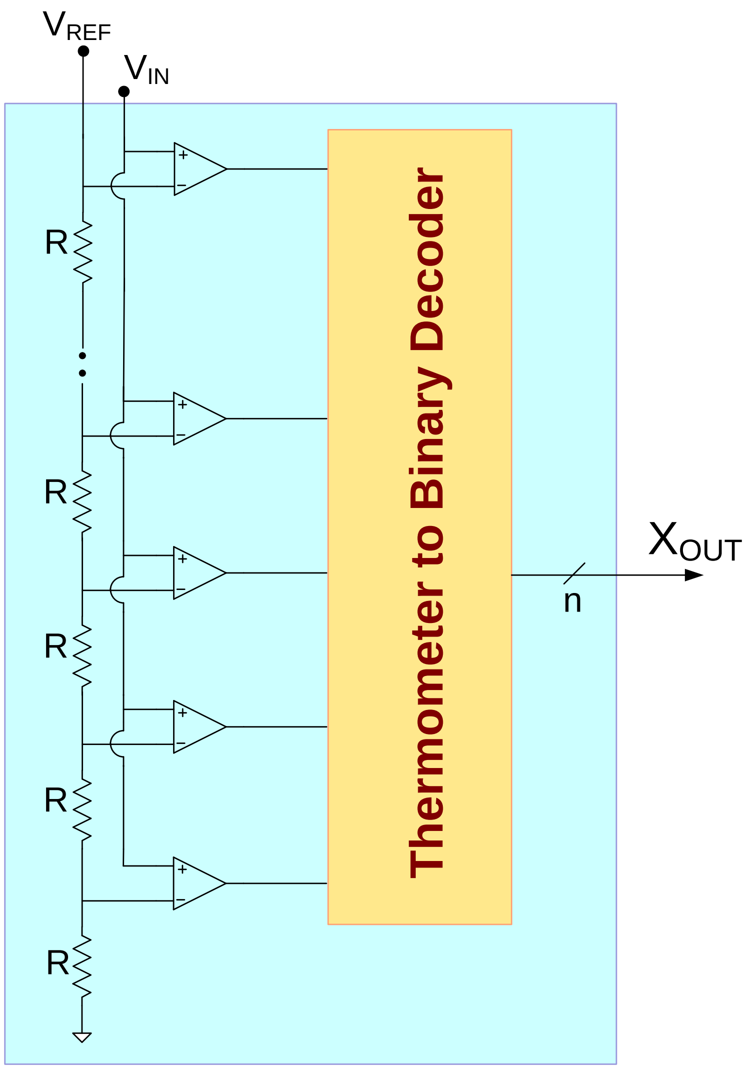
Statistical Characterization of Electronic Components and Circuits

Recall: Almost all data converter structures work perfectly if components are ideal

Major challenges in data converter design

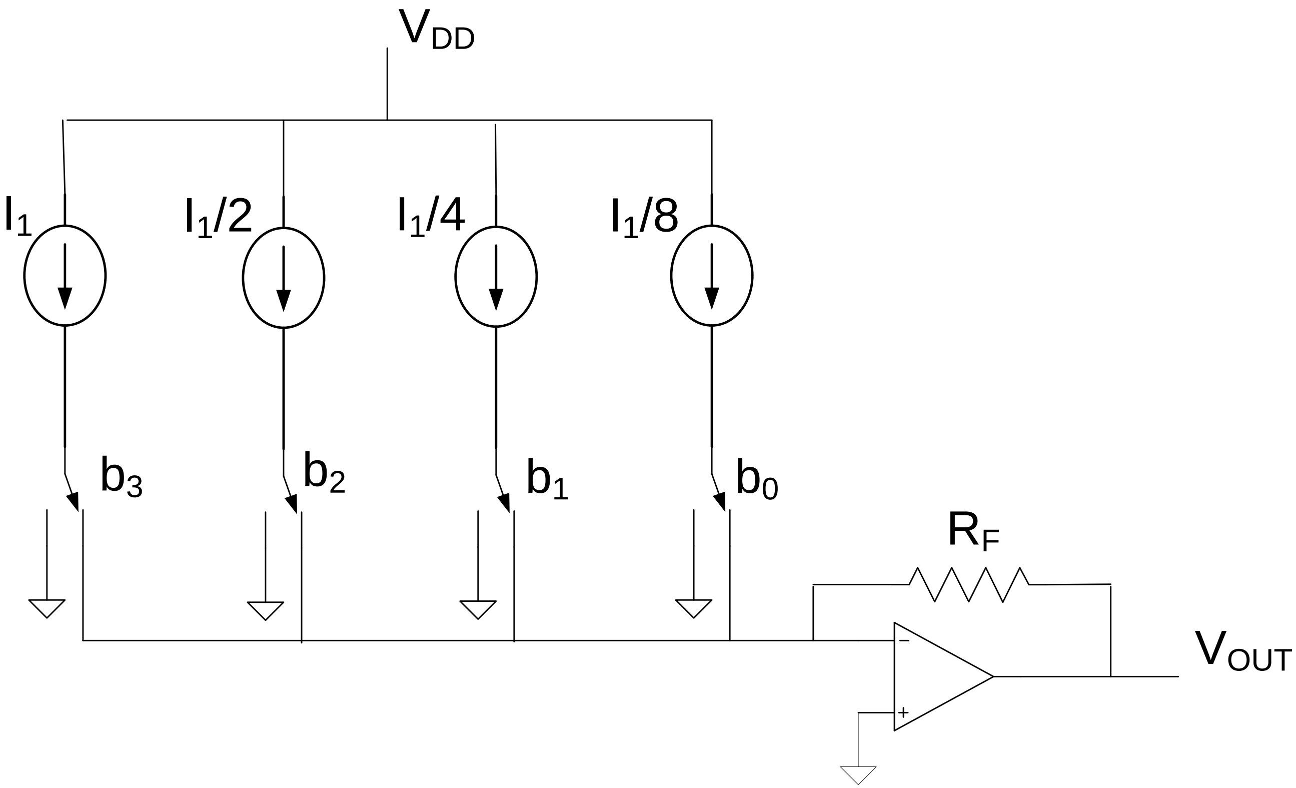
- Parasitic Resistances and Capacitances
- Nonlinearity in components
- Statistical variation in components and circuits
- Model uncertainties
- Power supply variability

Consider a flash ADC



- Resistor values and offset voltages of Comparators are all random variables at design level
- Variations of these RVs affect the break point and thus the yield

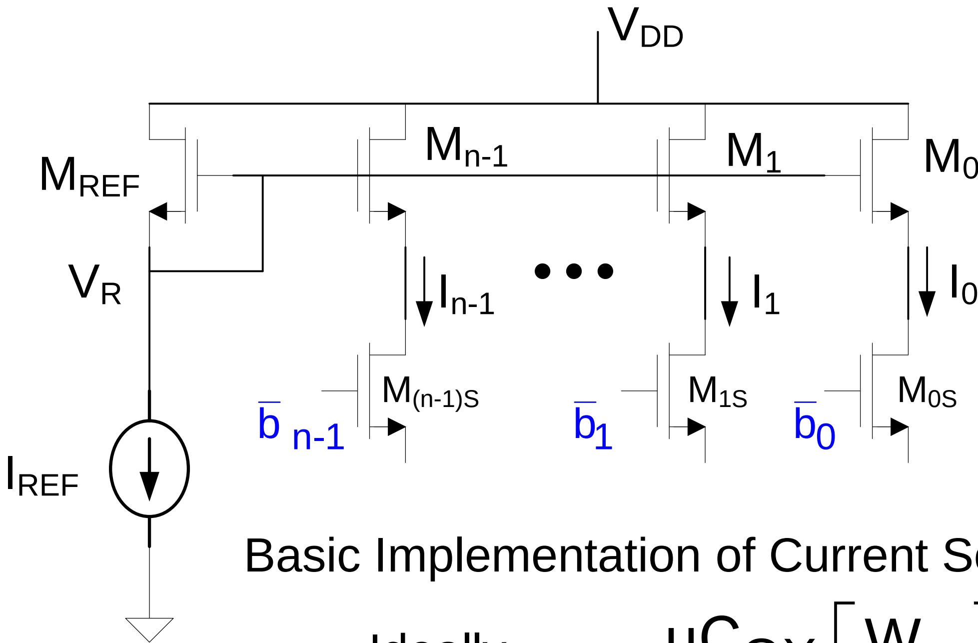
Consider Current-Steering DAC

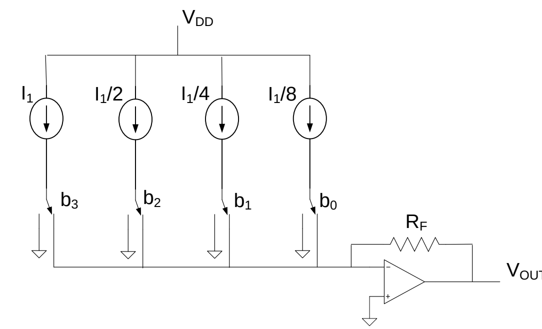


Ideally

$$V_{OUT} = -I_1 \bullet R_F \sum_{i=0}^{n-1} \frac{b_i}{2^{n-i}}$$

Consider Current-Steering DAC





Basic Implementation of Current Sources

Ideally $$I_m = \frac{\mu C_{OX}}{2}\left[\frac{W_m}{L_m}\right]\left(V_R - V_{Tp}\right)^2$$

$L_m = L_0$

$W_m = 2^{m-1} W_0$

Actually $$I_m \cong \frac{\mu_k C_{OXk}}{2}\left[\frac{W_{m_k}}{L_{m_k}}\right]\left(V_R - V_{Tpk}\right)^2$$

I_m is a random variables and is a function of the model parameters μ_k, C_{OXk}, W_{mk}, L_{mk}, and V_{Tpk}

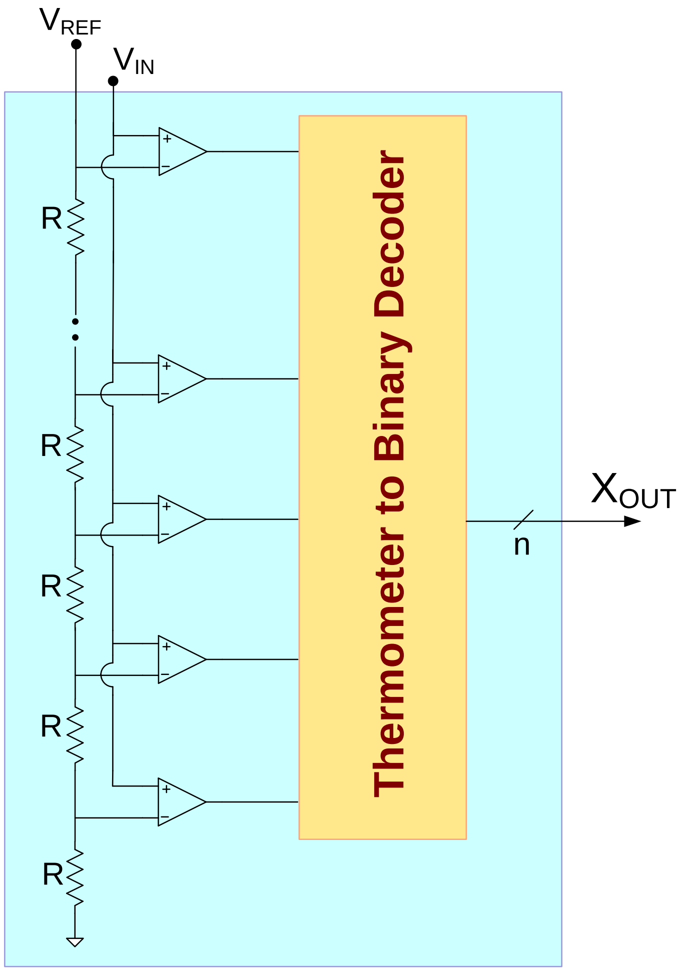
μ_k, C_{OXk}, W_{mk}, L_{mk}, and V_{Tpk} are all random variables

Recall from previous lecture

How important is statistical analysis?

Example: 7-bit FLASH ADC with R-string DAC

Assume R-string is ideal, V_{REF}=1V and V_{OS} for each comparator must be at most +/- ½ LSB



Case 1

Standard deviation is 5mV

$$P_{COMP} = 0.565$$

$$Y_{ADC} = 3.2 \bullet 10^{-32}$$

Case 2

Standard deviation is 1mV

$$P_{COMP} = 0.999904$$

$$Y_{ADC} = 0.988$$

Statistics play a key role in the performance and consequently yield of a data converter

Statistical Analysis Strategy

Will first focus on statistical characterization of resistors, then extend to capacitors and transistors

Every resistor R can be expressed as

$$R=R_N+R_{RP}+R_{RW}+R_{RD}+R_{RGRAD}+R_{RL}$$

where R_N is the nominal value of the resistor and the remaining terms are all random variables

R_{RP}: Random process variations
R_{RW}: Random wafer variations
R_{RD}: Random die variations
R_{RGRAD}: Random gradient variations
R_{RL}: Local Random Variations

- Data Converters (ADCs and DACs) are ratiometric devices and performance often dominated by ratiometric device characteristics (e.g. matching)

- Many other AMS functions are dependent upon dimensioned parameters and often not dependent upon matching characteristics

Statistical Analysis Strategy

$$R=R_N+R_{RP}+R_{RW}+R_{RD}+R_{RGRAD}+R_{RL}$$

R_{RP}: Random process variations
R_{RW}: Random wafer variations
R_{RD}: Random die variations
R_{RGRAD}: Random gradient variations
R_{RL}: Local Random Variations

$$\sigma_{RP} >> \sigma_{RW} >> \sigma_{RD}$$

- All variables globally uncorrelated
- For good common-centroid layouts gradient effects can be neglected
- Local random variations often much smaller than R_{RP}, R_{RW}, and R_{RD} though not necessarily
- Area dominantly determines σ_{RL}, but area has little effect on the other variables
- At the resistor-level on a die, R_{RP}, R_{RW} and R_{RD} highly correlated thus cause no mismatch
- Major challenge in data converter design is managing R_{RL} effects
- All zero mean and approximately Gaussian (truncated)
- For dimensioned performance characteristics (e.g. band edge of filter), R_{RP}, R_{RW} and R_{RD} are dominant and R_{RGRAD} and R_{RL} typically secondary

For notational convenience, assume $R=R_N+R_R$

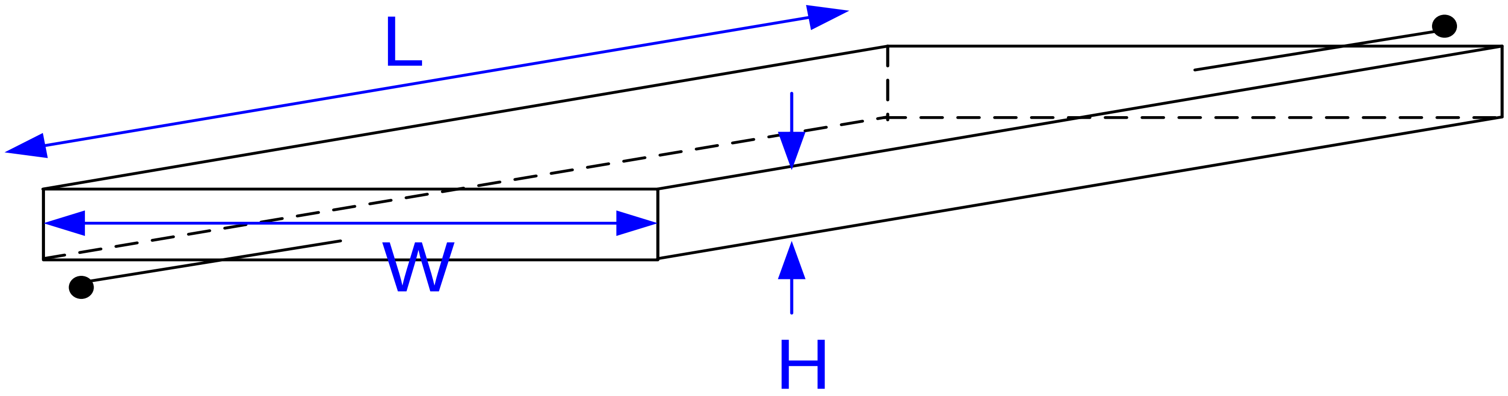
R_N includes R_{RP}, R_{RW} and R_{RD}, R_{GRAD} neglected, $R_R=R_{RL}$

Stay Safe and Stay Healthy !

End of Lecture 8

Resistor Characterization

Resistors are generally made of thin films of conductive or semiconductor materials



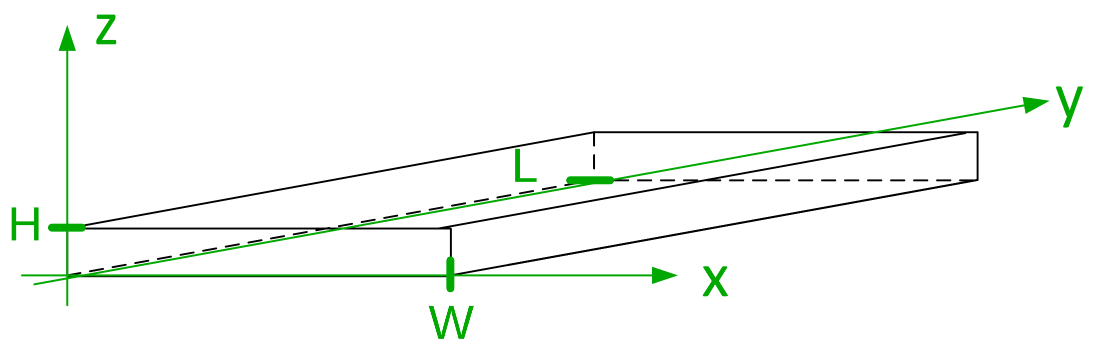
Generally h is very small compared to L and W

Films are often characterized by Sheet Resistance

In the ideal case $$R=\rho\left(\frac{1}{H}\bullet\frac{L}{W}\right)=R_{\square}\left(\frac{L}{W}\right)$$

Resistor Characterization

Resistors are generally made of thin films of conductive or semiconductor materials



Film Characterized by Resistivity : $\rho(x,y,z)$

Films are often characterized by Sheet Resistance $R_{\square}(x,y)=\dfrac{\rho(x,y,z)}{H(x,y)}$

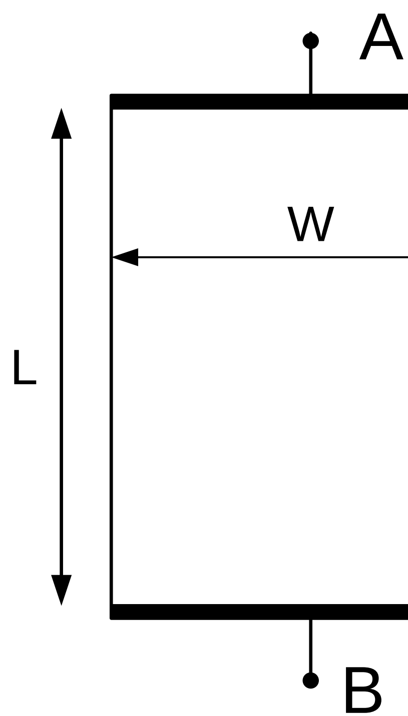
Ideally ρ(x,y,z) is independent of position as is $R_{\square}(x,y)$

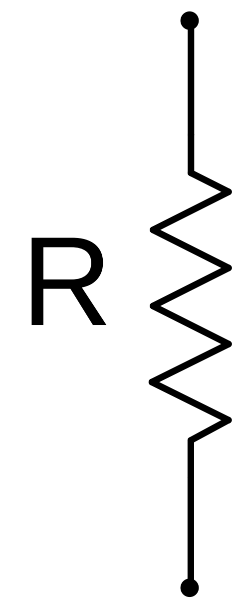
In the ideal case $R=\rho\left(\dfrac{1}{H}\bullet\dfrac{L}{W}\right) = R_{\square}\left(\dfrac{L}{W}\right)$

Resistor Characterization

Resistors are generally made of thin films of conductive or semiconductor materials

Ideally

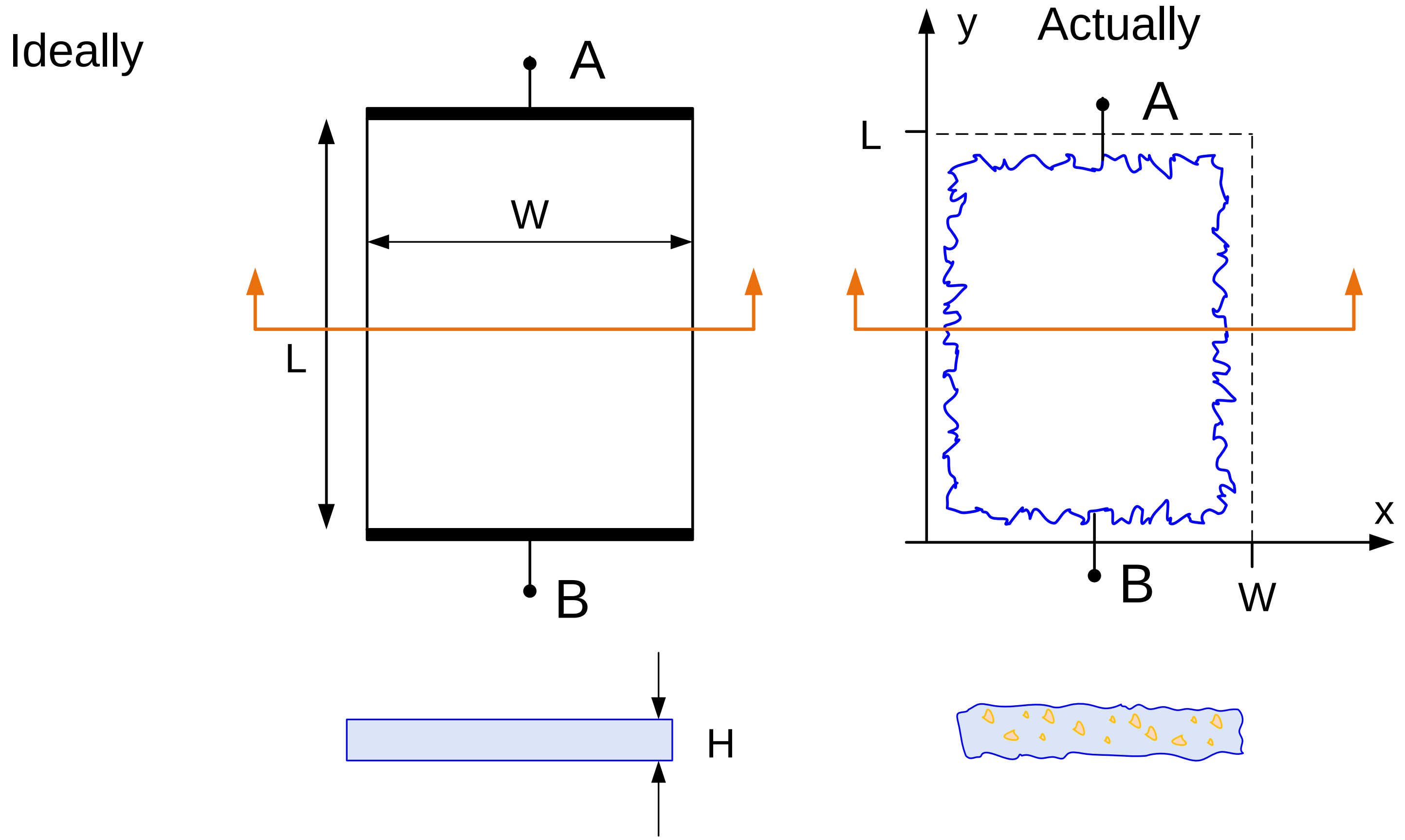




$$R = R_{\square}\left(\frac{L}{W}\right)$$

$$R_{\square} = \frac{\rho}{h}$$

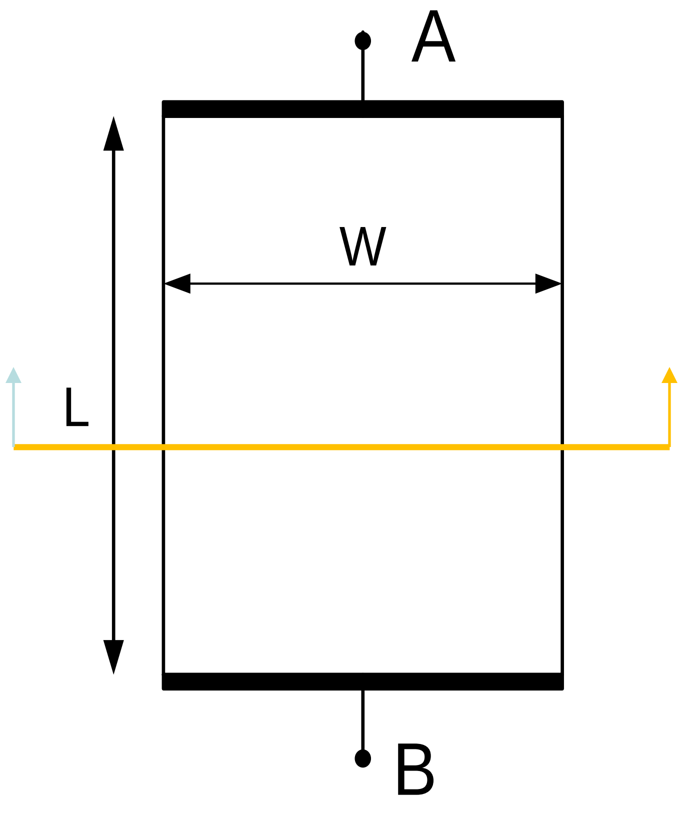
Resistor Characterization



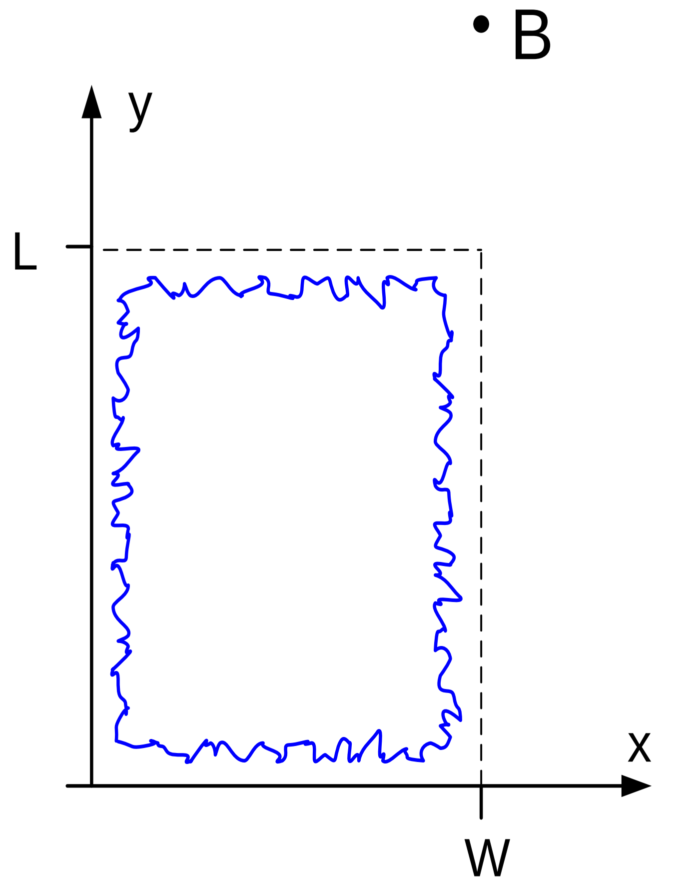
- Boundary of resistor varies with position
- ρ(x,y,z) varies with position
- Thickness (H(x,y)) varies with position
- Properties of resistor vary with position and temperature

Resistor Characterization

Ideally



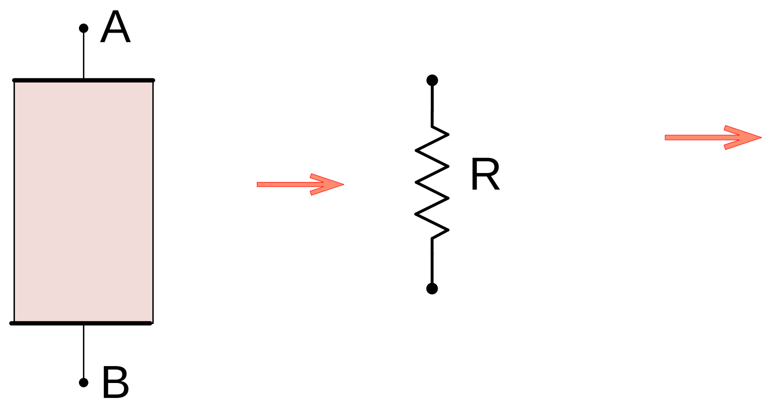
Actually



Boundary of resistor varies
ρ(x,y,z) varies with position

These variations will define R_R

Consider the following resistor circuits



$R = R_N + R_R$

Statistical Model

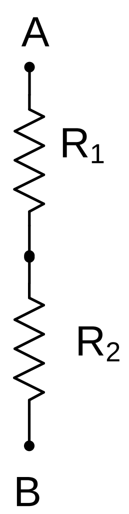
mean $\mu_{R_R} = 0$

standard deviation σ_{R_R}

Distribution: Truncated Gaussian

$$N \sim \left(0, \sigma_{R_R}\right)$$

Series Resistor Connection (of two nominally identical devices)



$R_1 = R_N + R_{R1}$

$R_2 = R_N + R_{R2}$

$R_{Ser\,2} = 2R_N + R_{R1} + R_{R2}$

Compare the standard deviation of the resistance of the series combination with that of a single resistor

Consider the following Theorem:

Theorem: If $X_1, \ldots X_n$ are uncorrelated random variables and $a_1, .. a_n$ are real numbers, then the random variable Y defined by

$$Y = \sum_{i=1}^{n} a_i X_i$$

has mean and variance given by

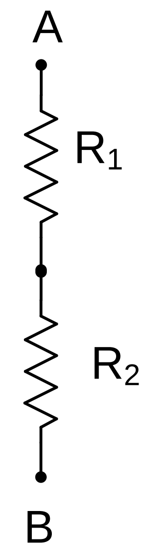
$$\mu_Y = \sum_{i=1}^{n} a_i \mu_i$$

$$\sigma_Y = \sqrt{\sum_{i=1}^{n} (a_i \sigma_i)^2}$$

where μ_i and σ_i are the mean and variance of X_i for i=1,…n.

Series Resistor Connection

(of nominally identical devices)

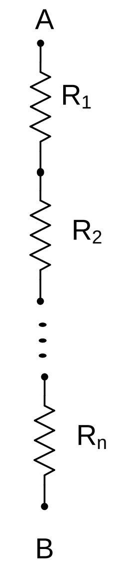


$$\left.\begin{array}{l} R_1 = R_N + R_{R1} \\ R_2 = R_N + R_{R2} \end{array}\right\} \quad R_{Ser\,2} = 2R_N + R_{R1} + R_{R2}$$

From Theorem $\quad \sigma_{Ser\,2} = \sqrt{2}\sigma_{R_R}$

$$N \sim \left(0, \sqrt{2}\sigma_{R_R}\right)$$

Extending to n-resistors that are nominally identical



$$R_{Sern} = nR_N + \sum_{k=1}^{n} R_{Rk}$$

$$\sigma_{Sern} = \sqrt{n}\sigma_{R_R}$$

$$N \sim \left(0, \sqrt{n}\sigma_{R_R}\right)$$

Summary of Results

Structure	Nominal Resistance	Standard Deviation	Normalized Standard Deviation
R	R_N	σ_{R_R}	
Ser nR	nR_N	$\sqrt{n}\sigma_{R_R}$	

Note increasing the resistance by a factor of n increased the standard deviation by $\sqrt{n}$

Normalized Statistical Characterization

$$\sigma_{\frac{R}{R_N}} = ?$$

From previous theorem:

For single resistor R

$$\sigma^2_{\frac{R}{R_N}} = \frac{1}{R_N^2}\sigma^2_{R_R} \quad \longrightarrow \quad \sigma_{\frac{R}{R_N}} = \frac{1}{R_N}\sigma_{R_R}$$

For series connection of n ideally identical resistors

$$\mathrm{R}_{EQ} = \mathrm{nR}_N + \sum_{k=1}^{n} \mathrm{R}_{\mathrm{Rk}}$$

$$\mathrm{R}_{\mathrm{EQNorm}} = \frac{\mathrm{R}_{EQ}}{\mathrm{nR}_{\mathrm{N}}} = \frac{\mathrm{nR}_N + \sum_{k=1}^{n} \mathrm{R}_{\mathrm{Rk}}}{\mathrm{nR}_{\mathrm{N}}} = 1 + \frac{1}{n}\sum_{k=1}^{n}\frac{\mathrm{R}_{\mathrm{Rk}}}{\mathrm{R}_{\mathrm{N}}}$$

$$\sigma^2_{\frac{R_{EQ}}{nR_N}} = \frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{R_N^2}\sigma^2_{R_R} = \frac{1}{n^2}\sum_{k=1}^{n}\sigma^2_{\frac{R_R}{R_N}} = \frac{1}{n}\sigma^2_{\frac{R_R}{R_N}} \quad \longrightarrow \quad \sigma_{\frac{R_{EQ}}{nR_N}} = \frac{1}{\sqrt{n}}\sigma_{\frac{R_R}{R_N}}$$

Note increasing the resistance by a factor of n dropped the normalized standard deviation by $\sqrt{n}$

Summary of Results

Structure	Nominal Resistance	Standard Deviation	Normalized Standard Deviation
R	R_N	$\sigma_R = \sigma_{R_R}$	$\sigma_{\frac{R_R}{R_N}}$
Ser nR	nR_N	$\sqrt{n}\sigma_{R_R}$	$\frac{1}{\sqrt{n}}\sigma_{\frac{R_R}{R_N}}$

Note increasing the resistance by a factor of n increased the standard deviation by $\sqrt{n}$

Note increasing the resistance by a factor of n decreased the normalized standard deviation by $\sqrt{n}$

Parallel Resistor Connection

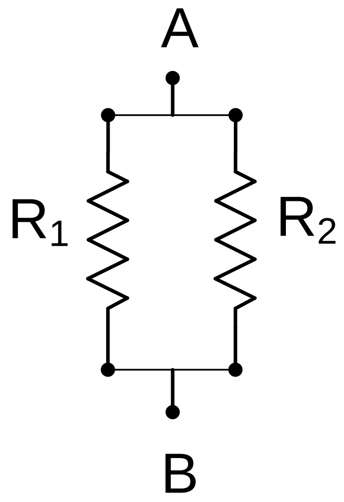
$$\left.\begin{aligned} R_1 &= R_N + R_{R1} \\ R_2 &= R_N + R_{R2} \end{aligned}\right\}$$

$$R_{Par\,2} = \frac{\left(R_N + R_{R_{R1}}\right)\left(R_N + R_{R_{R2}}\right)}{2R_N + R_{R_{R1}} + R_{R_{R2}}}$$

$$R_{Par\,2} = \frac{R_N^2 + R_N\left(R_{R_{R1}} + R_{R_{R2}}\right) + R_{R_{R1}} R_{R_{R2}}}{2R_N + R_{R_{R1}} + R_{R_{R2}}}$$

$$R_{Par\,2} \cong \frac{R_N^2}{2R_N} \frac{1 + \dfrac{R_{R_{R1}} + R_{R_{R2}}}{R_N}}{1 + \dfrac{R_{R_{R1}} + R_{R_{R2}}}{2R_N}}$$

$$R_{Par\,2} \cong \frac{R_N}{2} \frac{1 + \dfrac{R_{R_{R1}} + R_{R_{R2}}}{R_N}}{1 + \dfrac{R_{R_{R1}} + R_{R_{R2}}}{2R_N}}$$

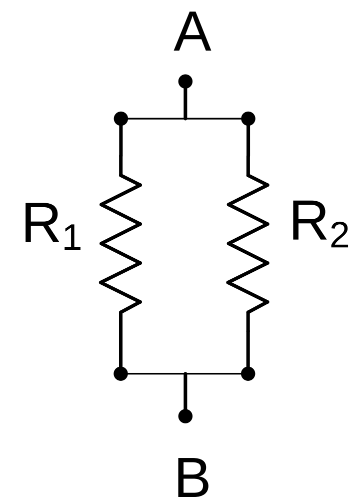


- The random variable R_{Par2} is highly nonlinear in R_{R1} and R_{R2}
- Some very good approximations of R_{Par2} can be made that linearize the expression

Parallel Resistor Connection

$$\left.\begin{aligned} R_1 &= R_N + R_{R1} \\ R_2 &= R_N + R_{R2} \end{aligned}\right\}$$

$$R_{Par\,2} \cong \frac{R_N}{2}\,\frac{1+\dfrac{R_{R_{R1}}+R_{R_{R2}}}{R_N}}{1+\dfrac{R_{R_{R1}}+R_{R_{R2}}}{2R_N}}$$



Recall that for x small, $\dfrac{1}{1+x} \cong 1-x$

Thus

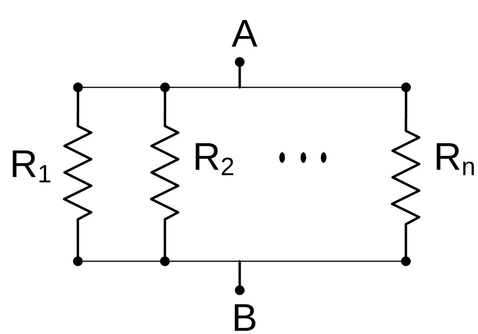
$$R_{Par\,2} \cong \frac{R_N}{2}\left(1+\frac{R_{R_{R1}}+R_{R_{R2}}}{R_N}\right)\left[1-\frac{R_{R_{R1}}+R_{R_{R2}}}{2R_N}\right] \cong \frac{R_N}{2}+\frac{1}{4}R_{R_{R1}}+\frac{1}{4}R_{R_{R2}}$$

From Theorem

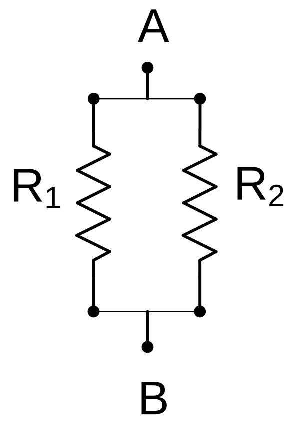
$$\sigma^2_{R_{Par\,2}} = \frac{1}{16}\sigma^2_{R_R} + \frac{1}{16}\sigma^2_{R_R} \cong \frac{1}{8}\sigma^2_{R_R} \quad\Rightarrow\quad \sigma_{R_{Par\,2}} \cong \frac{1}{\sqrt{8}}\sigma_{R_R}$$

For n in parallel, it follows that

$$\sigma_{R_{Parn}} \cong \frac{1}{n^{3/2}}\sigma_{R_R}$$



Parallel Resistor Connection



Consider normalized variance

$$R_{Par-2} = \frac{R_N}{2}$$

$$\frac{R_{Par\,2}}{R_{Par\,2-Norm}} \cong 1 + \frac{1}{2}\frac{R_{R_{R1}}}{R_N} + \frac{1}{2}\frac{R_{R_{R2}}}{R_N}$$

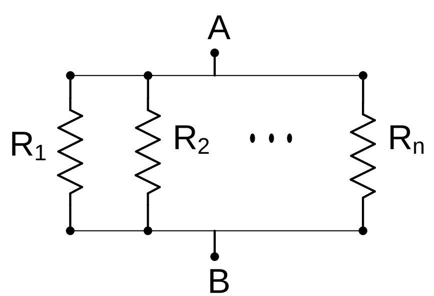
From Theorem

$$\sigma^2_{\frac{R_{Par\,2}}{R_{Par\,2-Norm}}} \cong \frac{1}{4}\sigma^2_{\frac{R_{R_{R1}}}{R_N}} + \frac{1}{4}\sigma^2_{\frac{R_{R_{R1}}}{R_N}} = \frac{1}{2}\sigma^2_{\frac{R_{R_{R1}}}{R_N}}$$

$$\sigma_{\frac{R_{Par\,2}}{R_{Par\,2-Norm}}} \cong \frac{1}{\sqrt{2}}\sigma_{\frac{R_{R_{R1}}}{R_N}}$$

And for n in parallel

$$R_{Par-n} = \frac{R_N}{n}$$



$$\sigma_{\frac{R_{Parn}}{R_{Parn-Norm}}} \cong \frac{1}{\sqrt{n}}\sigma_{\frac{R_R}{R_N}}$$

Note decreasing the resistance by a factor of n dropped the standard deviation by $\sqrt{n}$

Summary of Results

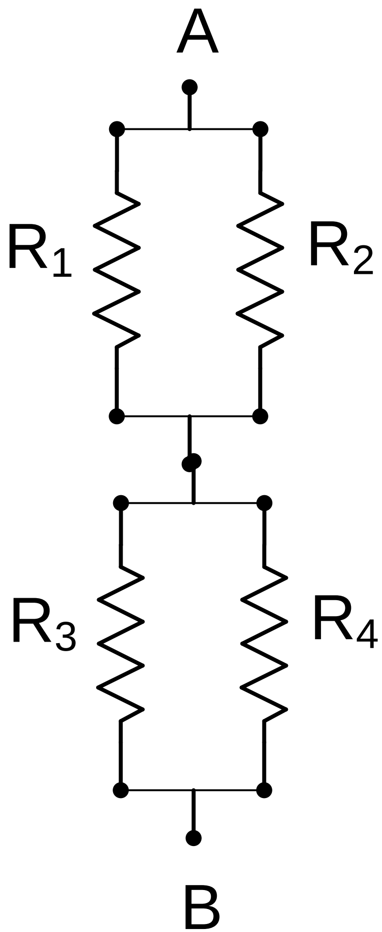
Structure	Nominal Resistance	Standard Deviation	Normalized Standard Deviation
R	R_N	$\sigma_R = \sigma_{R_R}$	$\sigma_{\frac{R_R}{R_N}}$
Ser nR	nR_N	$\sqrt{n}\,\sigma_{R_R}$	$\frac{1}{\sqrt{n}}\sigma_{\frac{R_R}{R_N}}$
Par nR	$\frac{R_N}{n}$	$\frac{1}{n^{3/2}}\sigma_{R_R}$	$\frac{1}{\sqrt{n}}\sigma_{\frac{R_R}{R_N}}$

Note increasing or decreasing the resistance by a factor of n decreased the normalized standard deviation by $\sqrt{n}$

Note increasing the area by a factor of n decreased the normalized standard deviation by $\sqrt{n}$

What is the relationship between resistance, area, and standard deviation?

Consider parallel/series combination of 4 nominally identical resistors



$$R_{EQ} = R_N$$

$$\sigma_{R_{EQ}} = \frac{\sigma_R}{2}$$

$$\sigma_{\frac{R_{EQ}}{R_N}} = \frac{1}{2}\sigma_{\frac{R}{R_N}}$$

Note making no change in the resistance reduced the standard deviation by 2

Note increasing the area by a factor of 4 dropped the standard deviation by 2

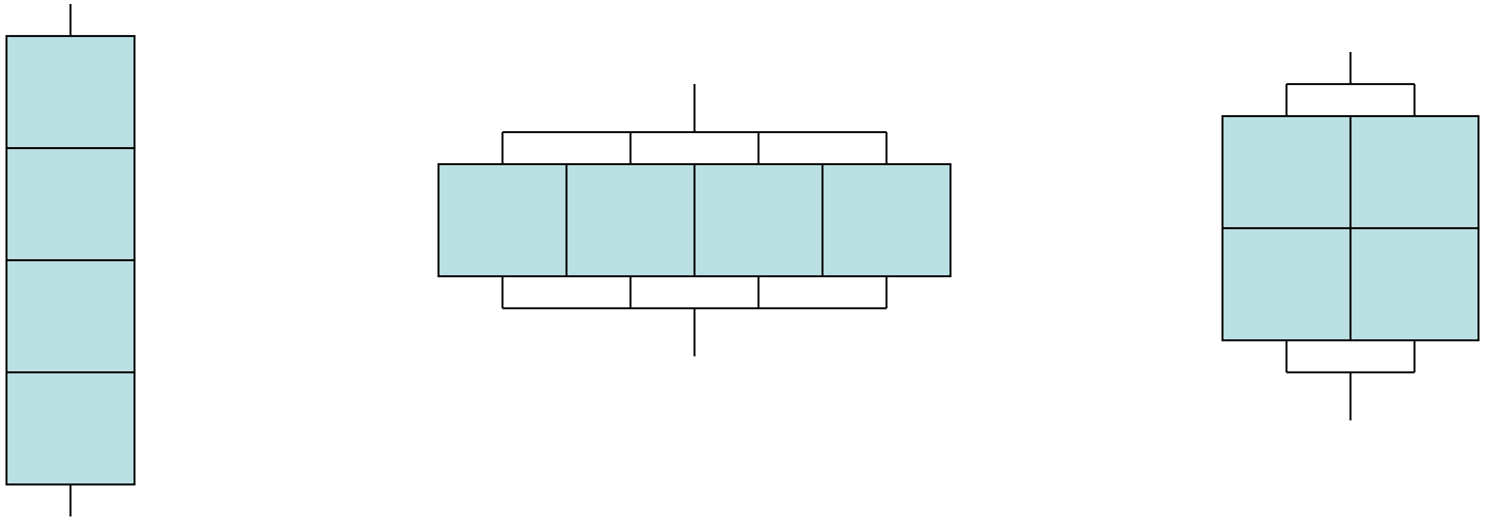
Summary of Results

Structure	Nominal Resistance	Standard Deviation	Normalized Standard Deviation
R	R_N	σ_{R_R}	$\sigma_{\frac{R_R}{R_N}}$
Ser nR	nR_N	$\sqrt{n}\sigma_{R_R}$	$\frac{1}{\sqrt{n}}\sigma_{\frac{R_R}{R_N}}$
Par nR	$\frac{R_N}{n}$	$\frac{1}{n^{3/2}}\sigma_{R_R}$	$\frac{1}{\sqrt{n}}\sigma_{\frac{R_R}{R_N}}$
Ser 2R	$2R_N$	$\sqrt{2}\sigma_{R_R}$	$\sigma_{\frac{R_R}{R_N}}/\sqrt{2}$
Par 2R	$\frac{R_N}{2}$	$\sigma_{R_R}/\sqrt{8}$	$\sigma_{\frac{R_R}{R_N}}/\sqrt{2}$
Ser 4R	$4R_N$	$2\sigma_{R_R}$	$\sigma_{\frac{R_R}{R_N}}/2$
Par 4R	$\frac{R_N}{4}$	$\sigma_{R_R}/8$	$\sigma_{\frac{R_R}{R_N}}/2$
Par/Ser 4R	R_N	$\sigma_{R_R}/2$	$\sigma_{\frac{R_R}{R_N}}/2$

Observation:

In all cases, increasing the area by a factor of n decreases the normalized standard deviation by sqrt (n)

Have considered in previous examples the following scenarios



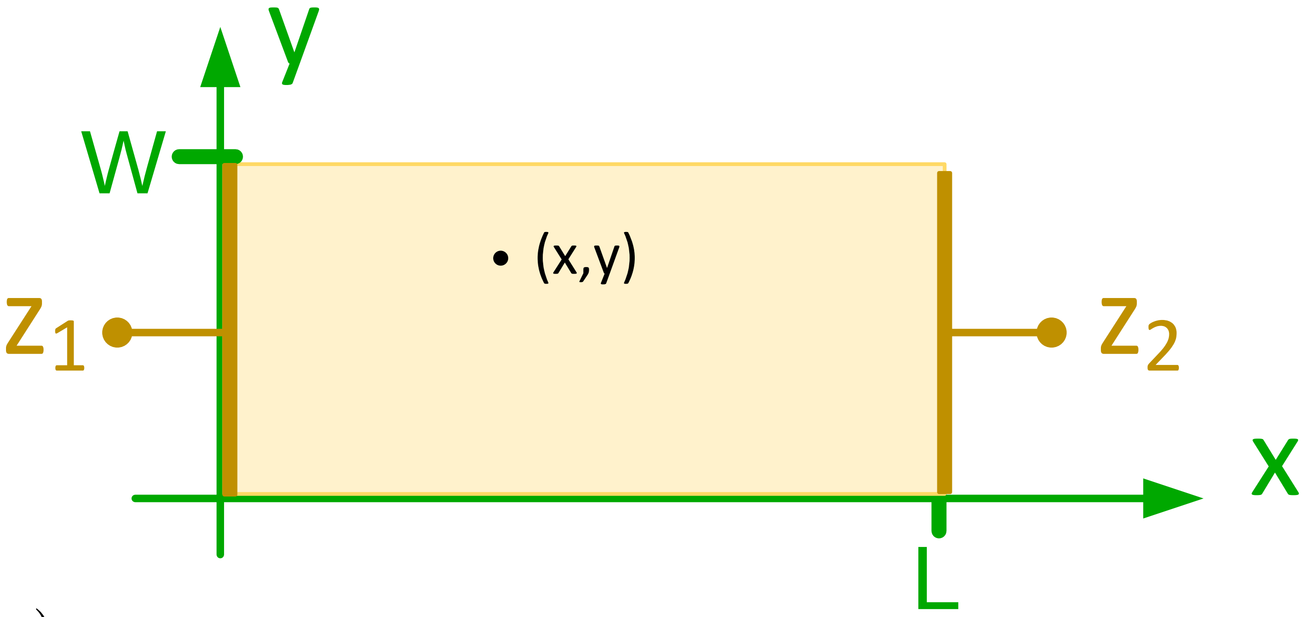
- Current density is uniform in each structure
- Aspect ratio plays no role in normalized performance
- Resistance value plays no role in normalized performance
- Only factor in normalized performance is area
- For a given resistance, each factor of 2 reduction in σ requires a factor of 4 increase in area

Key Implications:

If yield of a data converter is determined by matching performance, then every bit increment in performance will require at least a factor of 2 reduction in σ and correspondingly a factor of 4 increase in the area for the matching critical components if the same yield is to be obtained.

Formalize Resistor Characterization Concepts

Assume lithography is perfect, no gradient effects, and no contact resistance



$R_{\square}(x,y)$:   Sheet resistance at (x,y)

Most authors assume:

$$R_{\square EQ} = \frac{\int_A R_{\square}(x,y)\,dxdy}{A}$$

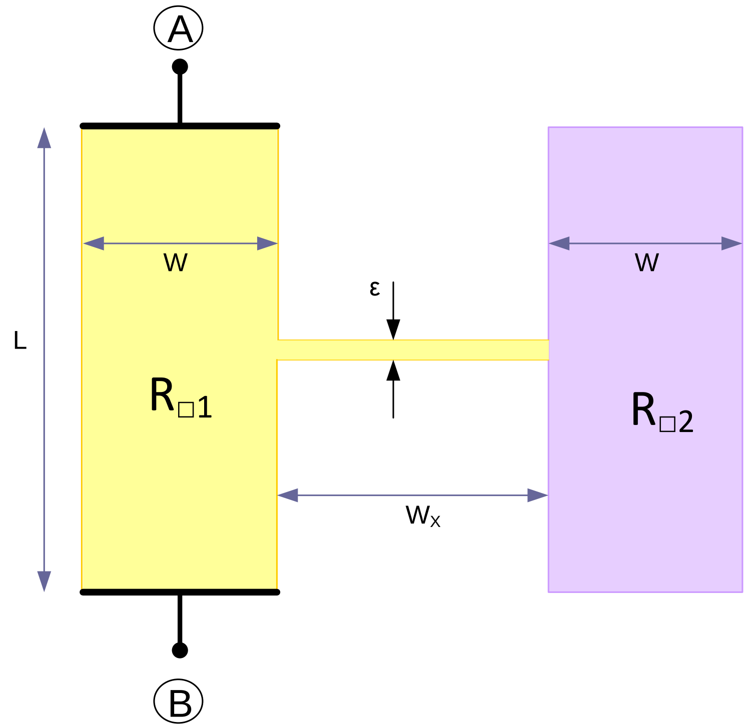
A=WL

$$R_{Z_1 Z_2} = R_{\square EQ}\frac{L}{W}$$

We will make this same assumption

Counter example showing limitations of standard assumptions

Assume sheet resistance constant in yellow region of value $R_{\square 1}$ and constant in purple region of value $R_{\square 2}$



If ε is small and W_X large $\quad R_{\square EQ} \cong R_{\square 1} \quad \Rightarrow \quad R_{AB} \cong R_{\square 1}\left(\frac{L}{W}\right)$

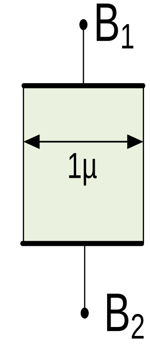
but

$$R_{\square EQ} = \frac{\dfrac{\int R_{\square}(x,y)\,dxdy}{A}}{A} \cong \frac{R_{\square 1}+R_{\square 2}}{2}$$

If $R_{\square 1}$ and $R_{\square 2}$ are not equal, then $R_{\square EQ} \neq R_{\square 1}$

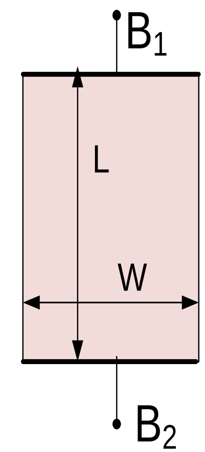
Though errors can be big, in practical processes the assumptions are probably pretty good !

Consider a square reference resistor of width 1μm



Assume the standard deviation of this reference resistor, <u>due to local random variations</u>, is σ_{REF}

Consider now a resistor of length L and width W



A=W•L

Define the equivalent sheet resistance of this resistor: $R_{\square EQ}$

$R_{\square EQ}$ is a random variable with a nominal value of $R_{\square N}$ and standard deviation that satisfies the expression

$$\sigma^2_{R_{\square EQ}} = \frac{\sigma^2_{REF}}{W \bullet L} = \frac{\sigma^2_{REF}}{A}$$

It follows that the value of the resistor R is given by the expression

$$R = R_{\square EQ} \bullet \frac{L}{W}$$

Thus

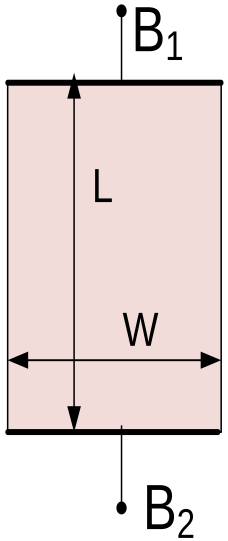
$$\sigma^2_R = \left(\frac{L}{W}\right)^2 \bullet \sigma^2_{R_{\square EQ}}$$

$$\sigma^2_R = \left(\frac{L}{W}\right)^2 \bullet \frac{\sigma^2_{REF}}{W \bullet L} = \sigma^2_{REF} \bullet \frac{L}{W^3}$$

Consider a resistor of width W and length L

$$\sigma_R^2 = \left(\frac{L}{W}\right)^2 \bullet \frac{\sigma_{REF}^2}{W \bullet L} = \sigma_{REF}^2 \bullet \frac{L}{W^3}$$

A=W•L



Consider now the normalized resistance $\dfrac{R}{R_N}$

where $\quad R_N = R_{\square N} \dfrac{L}{W}$

It follows that

$$\sigma^2_{\frac{R}{R_N}} = \left(\frac{1}{R_N^2}\right)\left(\sigma_{REF}^2 \frac{L}{W^3}\right) = \left(\frac{W^2}{R_{\square N}^2 L^2}\right)\left(\sigma_{REF}^2 \frac{L}{W^3}\right) = \left(\frac{1}{WL}\right)\left[\frac{\sigma_{REF}^2}{R_{\square N}^2}\right]$$

The term on the right in [ ] is the ratio of two process parameters so define the process parameter A_R by the expression $\quad A_R = \dfrac{\sigma_{REF}}{R_{\square N}}$

A_R is more convenient to use than both σ_{REF} and $R_{\square N}$

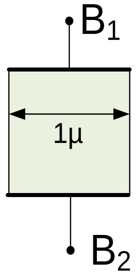
Thus the normalized resistance is given by the expression

$$\sigma^2_{\frac{R}{R_N}} = \frac{A_R^2}{WL} = \frac{A_R^2}{A}$$

Will term A_R the "Pelgrom parameter" (though Pelgrom only presented results for MOS devices)

How can A_R be obtained?

Recall: $$\sigma_{\frac{R}{R_N}} = \frac{A_R}{\sqrt{A}} \quad \text{where} \quad A_R = \frac{\sigma_{REF}}{R_{\square N}}$$



1. Obtain A_R from a PDK
2. Build a test structure to obtain A_R

Case 1

(How about this?)

1) Take a large number, n, of test resistors with length and width equal to 1μ

2) Measure $R_1, R_2, \cdots R_n$

3) Calculate the sample standard deviation

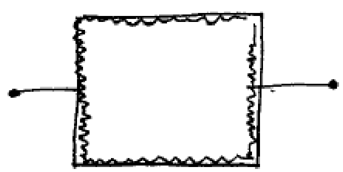
$$\hat{\sigma}_{\text{REF}} \cong \sigma_{\text{SAMPLE}}$$

$$\hat{R}_{\square N} \cong \mu_{\text{SAMPLE}}$$

$$A_R = \frac{\sigma_{REF}}{R_{\square N}}$$

$$A_R \cong \frac{\sigma_{SAMPLE}}{\mu_{SAMPLE}}$$

There are some serious problems with this approach!

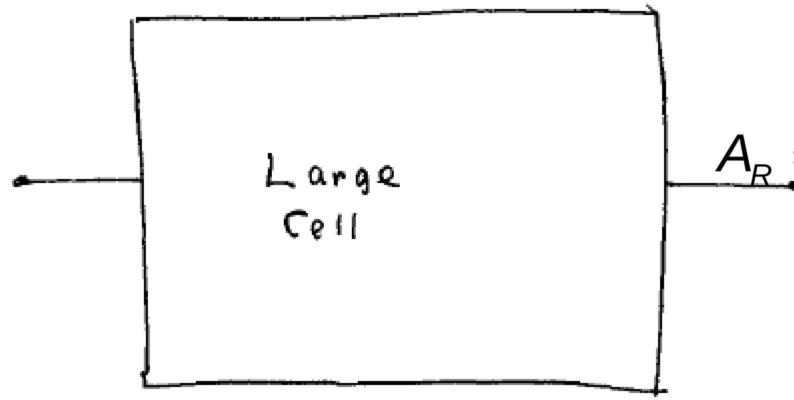


fringe effects will significantly skew $\hat{\sigma}_{\text{REF}}$

– increasing size can reduce/minimize this concern

If devices are not really close, other random variations including gradients will skew results that are supposed to characterize local random variations

Case 2



$$A_R = \frac{\sigma_{REF}}{R_{\square N}}$$

$$\widehat{R}_{\square N} \cong \frac{W}{L}\mu_{\text{SAMPLE}}$$

μ_{SAMPLE} is the mean resistance of the sample

$$\widehat{A}_R = \frac{\sigma_R\sqrt{LW}}{\mu_{SAMPLE}}$$

$$\sigma_R^2 = \sigma_{REF}^2 \bullet \frac{L}{W^3}$$

This strategy significantly reduces the boundary problem associated with the 1μ x 1μ structure

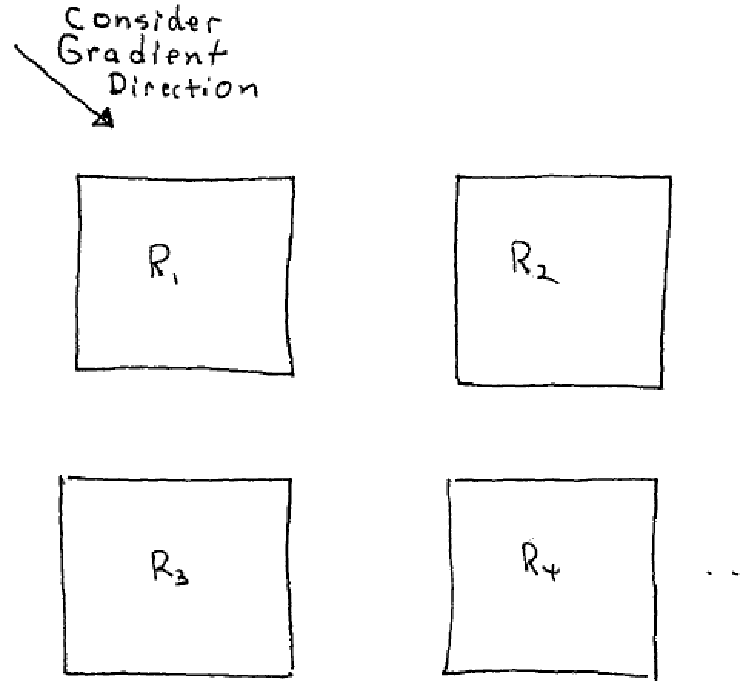
– but, this approach still has significant problems

Gradient effects will be particularly significant for large cells !

If devices are not really close, other random variations will skew results that are supposed to characterize local random variations

Gradient Effects

σ_{REF}

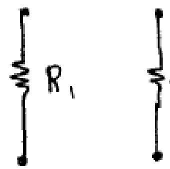


gradient effects will dramatically skew A_ρ extraction !

- need large test structures that are insensitive to gradient effects !
- consider a two-resistor test cell

How does the ratio matching of two resistors relate to the standard deviation of a single resistor?

$\longrightarrow \quad \sigma_R$ or $\sigma_{\frac{R}{R_N}}$



$R_{1N} = R_{2N} = R_N$

$$\Theta = \frac{R_1 - R_2}{R_N}$$

$$= \frac{R_N + R_{1R} - R_N - R_{2R}}{R_N}$$

$$\Theta = \frac{R_{1R} - R_{2R}}{R_N}$$

$$\therefore \sigma_\Theta^2 = \frac{1}{R_N^2}\left(\sigma_{R_{1R}}^2 + \sigma_{R_{2R}}^2\right)$$

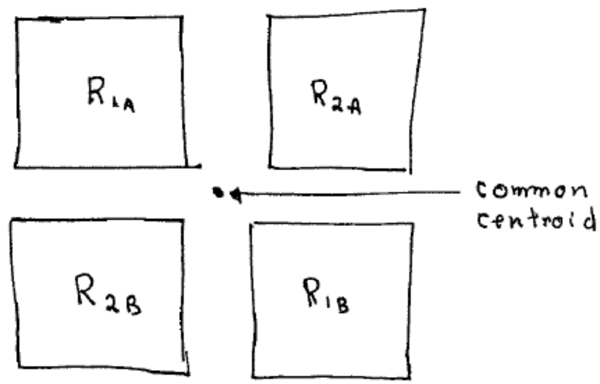
$$\sigma_\Theta^2 = \frac{2\sigma_{R_R}^2}{R_N^2} \longrightarrow \sigma^2_{\frac{\Delta R}{R_N}} = 2\sigma^2_{\frac{R}{R_N}}$$

Case 3 Measurement of A_R

$$\sigma_{\frac{\Delta R}{R_N}} = \sqrt{2}\,\sigma_{\frac{R}{R_N}}$$

$$A_R = \sqrt{A} \bullet \sigma_{\frac{R}{R_N}}$$

Strategy for test structures



A=area of one resistor

- large cells but not too big to create nonlinear gradients
- spread a large number of these test structures on a die
- generate $\frac{\Delta R_1}{R_N}, \frac{\Delta R_2}{R_N}, \ldots \frac{\Delta R_k}{R_N}$
- calculate variance of these samples $\hat{\sigma}_{\frac{\Delta R}{R_N}}$

$$\hat{A}_R = \sqrt{A} \bullet \sigma_{\frac{R}{R_N}} = \sqrt{A} \bullet \frac{1}{\sqrt{2}}\sigma_{\frac{\Delta R}{R_N}} = \sqrt{A} \bullet \frac{1}{\sqrt{2}}\hat{\sigma}_{\frac{\Delta R}{R_N}}$$

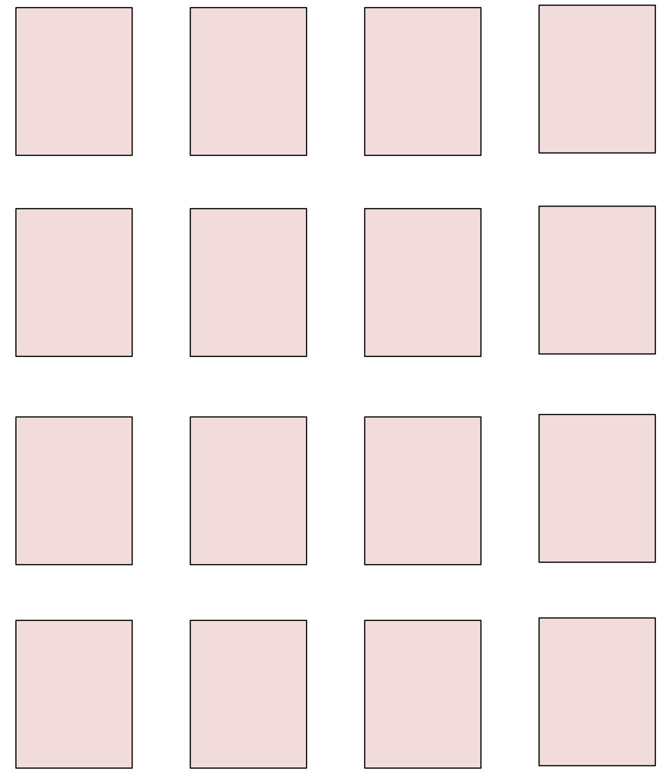
Measurement of A_R

$$\sigma_{\frac{R}{R_N}} = \frac{1}{\sqrt{2}}\sigma_{\frac{\Delta R}{R_N}} \cong \frac{1}{\sqrt{2}}\hat{\sigma}_{\frac{\Delta R}{R_N}}$$

$$\sigma_{\frac{R}{R_N}} = \frac{A_R}{\sqrt{A}}$$

$$A_R = \sqrt{\frac{A}{2}}\,\hat{\sigma}_{\frac{\Delta R}{R_N}}$$

Measurement of A_R



What about just taking a large number of resistors at multiple sites on a die, at multiple die locations on a wafer, and and on many wafers an wafer lots:

$$\left.\begin{aligned} \sigma_{\frac{R}{R_N}} &\cong \hat{\sigma}_{\frac{R}{R_N}} \\ \sigma_{\frac{R}{R_N}} &= \frac{A_R}{\sqrt{A}} \end{aligned}\right\} \qquad A_R = \sqrt{A}\,\hat{\sigma}_{\frac{\Delta R}{R_N}}$$

Stay Safe and Stay Healthy !

End of Lecture 8